Hone the Neoclassical Lens and Zoom in on Germany's Fiscal Stimulus Program 2008-2009

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Abstract

Business Cycle Accounting (BCA) by Chari, Kehoe, and McGrattan (2007, Econometrica) completes the "...through the lens of a neoclassical model"-approach. This paper refines and extends the methodology in four primary dimensions, creating a manual. i) the choice of the level of aggregation is critical and thus must be case-dependent. ii) a strict distinction between growth and cycle is beneficial. iii) BCA requires Maximum-Likelihood, even if it is difficult. Given these difficulties, we introduce a procedure that reliably and quickly locates the maximum and enables a detailed evaluation of the likelihood function and robustness checks. iv) it is revealing to discuss the results in the context of economic and political events. To illustrate the necessity and benefits of the refinements, we apply BCA to the Great Recession in Germany. The main driver was efficiency, followed by net exports and distortions in the markets for business investments. Government consumption and durable consumption acted counter-cyclically. We attribute the latter to a high subsidy for new cars or, more generally, for durables.

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1 Introduction

Chari, Kehoe, and McGrattan (2007) (CKM) complete the "...through the lens of a neoclassical model"-approach by designing Business Cycle Accounting (BCA).[1] For good reasons, BCA became a popular method with over a hundred applications in quantitative macroeconomics. Despite – or because of – this multiplicity, there is a lack of a precise, standardized procedure to prevent flawed implementations. To address that issue, we first extend BCA and guide it by reviewing all the basic steps in quantitative macro: data preparation, modeling, estimation, quantification, and discussion. The main contribution is a novel strategy for reliable and fast likelihood maximization and generally is a honed neoclassical lens.To exemplify the necessity and benefits of the refinements in all steps, we subsequently apply BCA to the Great Recession in Germany and the related, partially unconventional policy measures enacted by the German government to counter it.

The elementary BCA prototype economy extends the benchmark Real Business Cycle (RBC) model by time-varying distortions in nearly every market. CKM construe the origins of these distortions, or wedges, as taxes, nominal and real frictions, changes in expectations, etc. The interpretation of these wedges is non-structural, yet the parameterization corresponds to ad-valorem taxes, productivity, or government spending. The wedges' driver is a reduced-form Markov process, commonly approximated by a stationary vector autoregression (VAR) with one lag. Using empirical time series, one can estimate the parameters of the VAR process and determine the states of the wedges. These determined wedges are fed back into the model one by one to assess the contribution of each wedge to the business cycle.

At the structural level, the BCA prototype economy lies between estimated Dynamic Stochastic General Equilibrium (DSGE) models with deep structural equations, parameters, and shocks and statistical models, such as VARs. Compared to the former class of models, the findings are less assumption-driven as the lean structure, and also the applicability of Maximum-Likelihood estimation (MLE) minimizes the number of assumptions required. However, compared to the latter class of models, the structural level of the prototype economy is sufficient to distinguish between market distortions and agents' responses. This structure makes BCA appropriate as a first step in characterizing forces and counterforces of a particular business cycle or phase (e.g., Larkin, 2021) and, in the spirit

[1] Solow (1957) established this long-lasting approach. To name but a few more recent applications: Kehoe and Prescott (2002), Ohanian (2010), Cho and Doblas-Madrid (2013), Karabarbounis (2014) or Hansen and Ohanian (2016). Brinca et al. (2020) recently surveyed the BCA literature in detail.

of the inventors, in guiding researchers where to introduce frictions into DSGE models to replicate a cycle or phase. This idea contrasts the standard to characterize business cycle patterns and the targets of the theoretical predictions of DSGE models – impulse response functions and second moments from statistical models. Thus, we aim generally to lower the barriers for BCA to the extent that it becomes a standard in stating stylized facts of a business cycle or phase and deciding on the next steps, e.g., structural modeling.

Basically, each BCA analysis includes the five steps of quantitative macro mentioned above, starting with data preparation, i.e., the choice of the aggregation level and trend removal. Concerning the level of aggregation, the benchmark BCA prototype economy consists of three goods – consumption, investments, and the sum of government consumption and net exports. While this high-level aggregation keeps the analysis simple, it requires that individual goods aggregated in a particular quantity form a homogeneous entity. However, we are not aware of any BCA application discussing whether this holds. Although, the aggregation level can be crucial as we reminisce referring to the RBC literature.

Removing trends becomes critical as, for BCA, stationarity is a necessary condition for a successful estimation. The benchmark BCA prototype economy features one common log-linear trend for all quantities. Thus, detrending all time series with a common trend, e.g., as CKM, is consistent with the model but ensures no stationarity as the empirical equivalents are not entirely consistent with the balanced growth hypothesis. Therefore, we suggest beginning with the Vogelsang and Franses (2005) test for common deterministic linear trend growth rates. In case the test rejects the balanced growth hypothesis, only subtracting out an individual linear trend for each time series ensures stationarity. We argue furthermore that other detrending methods are inappropriate in the BCA context.

In the modeling step, the guiding principle is the consistency of the prototype economy with the data preparation. Previous work already extends the benchmark prototype economy in various ways, e.g., Šustek (2011) includes an asset market and a monetary policy wedge. Similarly, we propose to include more disaggregated markets, if necessary. Further, in the absence of balanced growth, i.e., if each time series were detrended separately, we suggest that the model should incorporate a long- and a short-run component of the wedges. The short-run component corresponds to the time-varying stochastic wedges, as introduced by CKM. The long-run component consists of a deterministic trend in the wedges, similar to Lu (2012) and del Río and Lores (2021), where the wedges grow deterministically, but they are time-varying. The wedges in our approach include both a deterministic, time-fixed growth rate component and a stochastic, stationary time-varying component to differentiate between growth and business cycle accounting.[2]

The estimation step determines the parameter values of the prototype economy and the states of the wedges using MLE. Applying MLE for BCA is difficult due to identification and numerical optimization problems. As a result, many avoid these problems by switching to Bayesian estimation.[3] However, as we argue, Bayesian methods are inappropriate for BCA – in short, it is impossible to make plausible prior assumptions for the parameter values of an approximated reduced-form process that describes fluctuations of non-structural wedges. Furthermore, Brinca et al. (2018) argue that weak identification associated with parameters of the VAR process is negligible, but unfortunately, is considerable for structural parameters. We introduce a two-step MLE procedure, which is fast and numerically robust. After verifying the strict local identification of all uncertain parameters using the strategy of Iskrev (2010), the procedure makes it feasible to plot the likelihood contour, detect the global likelihood maximum, and execute robustness checks to overcome weak identification problems concerning the uncertain structural parameters. Nevertheless, we recommend calibrating structural parameters whenever possible.

The two-step MLE procedure can be summarized as follows: In the first step, we maximize the likelihood function, which we receive from a Kalman recursion, assuming that the initial states are fixed and known in their long-run equilibrium. We show that this initialization is eventually equivalent to the procedure of CKM but provides two advantages, i) the computation of the likelihood function is less time-consuming and, ii) there exists an analytical and unique solution for the maximizing conditional covariance matrix. Under fairly general conditions, the estimator obtained with this procedure retains the properties of a Maximum-Likelihood (ML) estimator, yet, it is less efficient than the ML estimator based on the commonly used unconditional likelihood. The first-step estimates are thus only the guess for the actual (second-step) estimation, which maximizes the unconditional likelihood function. We complete the estimation step by determining the wedges' states with Kalman-smoothing.

CKM illustrate the impact of a wedge by plotting the path of economic activity if all but this wedge had remained unchanged since a reference year. We define additionally a

[2] This approach is similar to Gomme and Rupert (2007) where in a RBC model all quantities grow at a constant but not necessarily common rate.

[3] To name but a few, Otsu (2010), Chakraborty and Otsu (2013), Plotnikov (2017), Gerth and Otsu (2018) Mishra and Chatterjee (2021), or Klein and Otsu (2021).

measure, which quantifies the contribution of each wedge to the output gap.[4] This measure is specially adequate to check the robustness of a pro- or counter-cyclical effect of a wedge with respect to the model's parameters.

The discussion step is at the core of any BCA analysis. CKM map different structural frictions towards the reduced-form wedges and show that their results are equivalent. The discussion focuses on those equivalent results to get an idea of the origin of a particular distortion and of where to introduce frictions in more structural models. We propose an additional discussion of the results against the background of market interventions, e.g., due to stabilization policy. Mulligan (2005) initiates such discussions by studying policy interventions as reduced-form errors of RBC models, and Kersting (2008) initiates the mapping of political measures, namely the 1980's U.K. labor market reforms, towards wedges inside the BCA framework. More general, Hansen et al. (2020) explore historical events through the lens of a neoclassical model.

To emphasize the benefits of disaggregation and mapping policy events towards wedges, we would like to refer to the insightful work of Cheremukhin et al. (2015, 2016). They disaggregate the production into two sectors (agriculture and non-agriculture) and map the effects of the policy cycle from "Maoists" and "Pragmatists" in China and from the transformation from Tsarist Russia to the Soviet Union, respectively, towards wedges.

Our application to the Great Recession in Germany, together with the related German government stabilization policy measures, exemplifies the necessity of the discussed refinements. The policy measures amounted to 82 billion € or 3.2 percent of GDP and intervened on different markets, e.g., by increasing government spending and expanding short-time work possibilities. Particularly noteworthy is the German cash for clunkers program, since this car subsidy affected one of Germany's core industries and was internationally incomparably large (5 Billion € or 0.2 percent of GDP).

To account for the German export dependency, the increased government consumption, and the cash for clunkers program, which is equivalent to a subsidy for new durables, we distinguish between government spending and net exports as well as between business investments and new durables. The corresponding prototype economy includes the following wedges: *government consumption, durables, investment, labor, net exports,* and *efficiency*. Except for the labor wedge, each of these wedges includes a long- and a short-run component, as the data rejects the balanced growth hypothesis. It is feasible with our

[4]The corresponding definition of potential Gross Domestic Product (GDP) is the amount along the path the economy would follow if all wedges were fixed at the reference year.

two-step procedure to estimate the 57 parameters corresponding to the wedges' process and two structural parameters and handle weak identification of the latter.

Our findings suggest that the efficiency wedge mainly drove the crises (62%), followed by the net exports (26%) and the investment wedge (19%). The government consumption wedge and the durables wedge acted counter-cyclically (about -5% each). Furthermore, the labor wedge contributed only 3% to the crisis but induced a fast recovery. These statistics are robust except for the investment wedge. We discuss the results against different market interventions, e.g., we attribute the counter-cyclicality of the durables wedge to the cash for clunkers program.

Existing BCA applications for the Great Recession in Germany by Brinca et al. (2016) and Gerth and Otsu (2018) suggest negligible effects of the investment wedge on the business cycle. Both treat durables and other investment goods as a composite. We obtain similar results, feeding back both wedges simultaneously into the model. The pro-cyclicality of the investment wedge and the counter-cyclicality of the durables wedge cancel each other out, which is why previous work potentially underrates the importance of the investment wedge and, consequently, equivalent financial frictions. These insights also challenge the literature addressing the German fiscal stimulus program in structural DSGE models without accounting for durables separately (e.g., Gadatsch et al., 2016; Drygalla et al., 2018). Finally, we contribute to the literature on durable net-tax changes as a tool of unconventional fiscal policy (e.g., Clemens and Röger, 2021; Bachmann et al., 2021).

The remainder of the paper reads as follows. In the first part, we refine BCA inside a manual. There, we first discuss data processing, second modeling, third the technical implementation, including the two-step MLE procedure, and lastly, the presentation of the results. In the second part, we present our application. Finally, the paper concludes.

2 Manual

2.1 Data processing

BCA is an empirical exercise, and therefore data processing is a necessary preliminary step to align empirical measurements with theoretical assumptions. Empirical business cycle research usually requires determining the degree of aggregation in line with the research question and removing trends for stationary variables. We will discuss both steps in the following two paragraphs.

Aggregation Most BCA applications employ the CKM prototype economy with one sector producing one homogeneous good that is used for consumption, investment, and a residual consisting of government consumption and net exports. On the one hand, this parsimony, similar to the benchmark RBC model, is a major advantage because it guarantees straightforward insights. On the other hand, there are solid arguments for the use of a lower aggregation level in RBC models. E.g., Greenwood and Hercowitz (1991) argue that a differentiation between household and business investments is meaningful when the taxation of market activity is taken into account or if their cycle behavior differs in general. The latter aims at the fact that the quantities must form a homogeneous entity to some degree. Thus, the RBC literature analyzes more disaggregated markets and production sectors, for example, by distinguishing durable goods or housing from investments in productive capital. In BCA, leaving different cyclical behavior of the subaggregates out of consideration becomes critical as a positive distortion in the market of a subaggregate and a negative distortion in the market of another subaggregate cancel each other out in the aggregate, leaving important market distortions undetected. As a good balance between parsimony and higher information content, we suggest the following rule for the degree of disaggregation: As much as necessary, as little as possible.

Trend removing There are two common methods for the separation of the growth and cyclical components in the literature. One is to simply remove a common (log-)linear trend (e.g., Chari et al., 2007; Brinca et al., 2016), the other is to use a more sophisticated filter, like the HP-filter (e.g., Brinca et al., 2020).

Both approaches are problematic. Concerning two-sided filters like the HP-filter, we are not aware of how to determine the wedges states as data consistency requires to applying the same filter to the model predictions, which excludes any recursive method. In addition, theoretically, the stationary wedges' process explains all components of a time series except the log-linear trend. Consequently, any filter beyond log-linear detrending causes an unnecessary loss of information.

Detrending time series with the same growth rate is consistent with the benchmark BCA prototype economy as balanced growth is present. However, using this method, MLE is often not feasible because the balanced growth hypothesis is usually empirically untenable, leading to a lack of stationarity. As a consequence, the estimation procedure for the parameters of the underlying stationary stochastic process fails. We suggest a test for common deterministic linear trend growth rates as stated by Vogelsang and Franses (2005), e.g., based on the GDP growth rate, or more elaborated, subject to the linear restriction that the trend growth rates of the time series are all equal in the manner of DeJong and Dave (2011, Ch. 6.1). If one cannot assume a common trend, we suppose to detrend each time series by its own growth rate obtained from a linear trend estimation. Additionally, we suggest verifying the stationarity of hours worked. Both procedures together guarantee stationarity. To ensure consistency with the model, we propose to include a long- and a short-run component in the wedges of the model if necessary.

Trend removal by differencing is uncommon in the BCA context and in the case of different average growth rates it even cannot be consistently modeled.

2.2 The prototype economy

The CKM benchmark BCA prototype economy relies on the benchmark RBC model. Three agents exist in this economy, an infinitely-lived representative household, a firm, and a government. Various wedges extend the model, which we classify into three different types of wedges. The first type distorts markets just like ad valorem taxes, e.g., on investment goods or labor income. The second distorts efficiency and corresponds to factor productivity. The last distorts the aggregated demand as an additive separable demand subaggregate residual like exogenous government consumption. The wedges follow a Markov process and given the states of the wedges as well as budget and resource constraints the household chooses streams of consumption and leisure and the firm capital and labor to maximize lifetime utility and profits, respectively. The government finances its expenditures by levying taxes on labor and investment. In the case where a lower level of aggregation is necessary, the model must be adapted appropriately. Appendix B provides the full analytic framework of the CKM benchmark prototype economy.

Business cycle component A Markov process drives the vector $\mathbf{z}_t$ of the stochastic time-varying components of the wedges, which in turn drive the entire fluctuations in the model. The Markov process is standardly approximated by a VAR(1)-process and reads

$$\mathbf{z}_{t+1} = \mathbf{\Pi}\, \mathbf{z}_t + \boldsymbol{\epsilon}_{t+1},\ \boldsymbol{\epsilon}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{\Sigma}). \tag{1}$$

We combine the stochastic time-varying components $\mathbf{z}_t$ with the steady-state components additively or multiplicatively subject to the type of wedges.

Growth component In the CKM benchmark prototype economy, there are two growth components. The population growth and technical progress. The latter determines the balanced growth rate of per capita quantities. Moreover, it is straightforward to model separate trend growth of any quantities via a deterministic trend in relative prices. Formal, the relative price of a good or input factor X in period t, P_{Xt}, evolves with $P_{Xt} = g_{P_X} P_{Xt-1}$. Following, the growth factor of quantity X, g_X, evolves with g/g_{P_x} where g is the growth factor of an arbitrary numeraire. In this way, relative prices represent the wedges' deterministic long-run component and therewith correspond to a growth accounting procedure. Since the cyclical component includes the steady-state component, detrended prices $P_{Xt}/g^t_{P_x}$ can be normalized to one.

Mapping Here, we briefly discuss the mapping from structural models into the prototype economy from CKM, Brinca et al. (2016), and various other authors. Consider sticky wages in terms that in the short run the current real wages are a geometric mean of the current marginal product of labor and the real wage of the previous period, while in the long run, the marginal product of labor equals the marginal rate of substitution between consumption and labor. In this case, a wedge occurs as a deviation from the long-run equilibrium, hence sticky wedges are mappable to a prototype economy with a labor market wedge. To understand the mapping of political events, consider e.g., an investment grant to stimulate aggregated demand. Assume the reciprocal of an ad valorem tax ($\frac{1}{1+\tau}$) represents the investment wedge and interpret the investment grant as a subsidy. In that case, it is straightforward to recognize that the grant increases the investment wedge. Similarly, consider an easing of monetary policy in form of a lower base rate. As refinancing becomes cheaper, for a given real rate of return, investment increases. Hence, monetary policy changes the intertemporal decision, which is reflected in a higher investment wedge. In this way, it is possible to map various policy measures. Nutahara and Inaba (2012) apply BCA for misspecified wedges and find they are able to approximate the true wedges and the corresponding response of the agents adequately.

2.3 Implementing the Business Cycle Accounting framework

After aligning the theory with the data, we turn to the technical implementation of the BCA methodology. To perform a BCA analysis, we have to master four steps: pinning down the parameter values, solving for the policy function, determining the state's realization, and evaluating the contribution of a combination of wedges.

In the first paragraph, we present a suitable solution method for BCA and introduce the notation of the solved model. In the next paragraph, we discuss two ways to determine the realization of the model's states and review the wedge decomposition proposed by CKM. Suggestions on the calibration exercises are presented afterward. The last paragraph gives a detailed description of our MLE procedure. There, we also discuss why MLE is the appropriate estimation method subject to BCA.

Solution To derive the policy function, we use a linear perturbation method. In detail, we apply the method of undetermined coefficients as Uhlig (1999) and Christiano (2002) describe to solve the log-linearized model. An exact description of how we solve the model is given in Appendix D. This solution method is advantageous with respect to the estimation procedure as illustrated below because it allows solving separately for the deterministic and stochastic parts of the solution. The solved model then can be written as

$$\mathbf{y}_t = \mathbf{L}_{\mathbf{x}}^{\mathbf{y}} \cdot \mathbf{x}_t + \mathbf{L}_{\mathbf{z}}^{\mathbf{y}} \cdot \mathbf{z}_t, \tag{2a}$$

$$\mathbf{c}_t = \mathbf{L}_{\mathbf{x}}^{\mathbf{c}} \cdot \mathbf{x}_t + \mathbf{L}_{\mathbf{z}}^{\mathbf{c}} \cdot \mathbf{z}_t, \tag{2b}$$

$$\mathbf{x}_{t+1} = \mathbf{L}_{\mathbf{x}}^{\mathbf{x}} \cdot \mathbf{x}_t + \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} \cdot \mathbf{z}_t, \tag{2c}$$

where the matrices $\mathbf{L}_{\mathbf{x}}^{\cdot}$ characterize the policy function of the deterministic part of the model's solution, while $\mathbf{L}_{\mathbf{z}}^{\cdot}$ describe the policy function of the stochastic part. The vectors $\mathbf{x}_t$, $\mathbf{y}_t$ and $\mathbf{c}_t$ collect the log-linearized model's endogenous states and it's observed and unobserved control variables, respectively. $\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}$ is assumed to be non-singular, which is essential for the determination of the states in the procedure of CKM as well as in our estimation strategy.

State determination and wedge decomposition To obtain the endogenous and exogenous states, $\mathbf{x}_t$ and $\mathbf{z}_t$ for all $t = 1, 2, \ldots, N$, CKM assume that the economy was in its growth equilibrium in the period $t = 0$, i.e., $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = \mathbf{0}$ and $\mathbf{z}_0 = \mathbf{0}$. In this case we can use (2a) and (2c) to obtain $\mathbf{x}_t$ and $\mathbf{z}_t$ as

$$\mathbf{x}_t = \mathbf{Z}_1\, \mathbf{y}_{t-1} + \mathbf{Z}_2\, \mathbf{x}_{t-1}, \qquad \mathbf{Z}_1 := \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^{-1},\ \mathbf{Z}_2 := \mathbf{L}_{\mathbf{x}}^{\mathbf{x}} - \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^{-1} \mathbf{L}_{\mathbf{x}}^{\mathbf{y}}, \tag{3a}$$

$$\mathbf{z}_t = \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^{-1} \left(\mathbf{y}_t - \mathbf{L}_{\mathbf{x}}^{\mathbf{y}}\, \mathbf{x}_t\right), \qquad \mathbf{x}_0 = \mathbf{0},\ \mathbf{y}_0 = \mathbf{0},\ \forall t = 1, 2, \ldots, N. \tag{3b}$$

However, if we are not willing to make this additional assumption, we can also estimate $\mathbf{x}_t$ and $\mathbf{z}_t$ by Kalman smoothing as

$$\mathbf{x}_t = \mathbb{E}\left[\mathbf{x}_t | \mathbf{y}_N, \ldots, \mathbf{y}_1\right], \tag{4a}$$

$$\mathbf{z}_t = \mathbb{E}\left[\mathbf{z}_t | \mathbf{y}_N, \ldots, \mathbf{y}_1\right], \qquad \forall t = 1, 2, \ldots, N. \tag{4b}$$

The latter way of determining $\mathbf{x}_t$ and $\mathbf{z}_t$ is more natural in the sense that the initial states are usually unknown.[5]

Finally, we apply the wedge decomposition proposed by CKM to assess the contribution of a wedge or a combination of wedges to the quantities of interest. In detail, imagine we want to perform our BCA analysis over the periods $p, p+1, \ldots, p+k$ with $p \geq 1$ and $p + k \leq N$. Then to evaluate the contribution of a combination of wedges, which we allow to fluctuate, we would simulate a counterfactual set of data

$$\tilde{\mathbf{z}}_t = \mathbf{S}\ \mathbf{z}_t + (\mathbf{I} - \mathbf{S})\ \mathbf{z}_p, \tag{5a}$$

$$\tilde{\mathbf{x}}_t = \mathbf{L}^{\mathbf{x}}_{\mathbf{x}} \cdot \tilde{\mathbf{x}}_{t-1} + \mathbf{L}^{\mathbf{x}}_{\mathbf{z}} \cdot \tilde{\mathbf{z}}_{t-1}, \tag{5b}$$

$$\tilde{\mathbf{y}}_t = \mathbf{L}^{\mathbf{y}}_{\mathbf{x}} \cdot \tilde{\mathbf{x}}_t + \mathbf{L}^{\mathbf{y}}_{\mathbf{z}} \cdot \tilde{\mathbf{z}}_t, \tag{5c}$$

$$\tilde{\mathbf{c}}_t = \mathbf{L}^{\mathbf{c}}_{\mathbf{x}} \cdot \tilde{\mathbf{x}}_t + \mathbf{L}^{\mathbf{c}}_{\mathbf{z}} \cdot \tilde{\mathbf{z}}_t, \qquad \tilde{\mathbf{x}}_{p-1} = \mathbf{x}_{p-1},\ \tilde{\mathbf{z}}_{p-1} = \mathbf{z}_{p-1},\ \forall\ t = p, p+1, \ldots, p+k, \tag{5d}$$

where is $\mathbf{S} = \operatorname{diag}\left(s_1, \ldots, s_{n_z}\right)$ with

$$s_i = \begin{cases} 1, & \text{if the } i\text{-th wedge is allowed to fluctuate,} \\ 0, & \text{else.} \end{cases}, \quad i = 1, \ldots, n_z. \tag{6}$$

Calibration In general, we suggest determining as many structural parameters of the prototype economy under consideration as possible by calibration, so that ideally only the parameters affecting the stochastic process need to be estimated using MLE. Calibration might help to avoid problems resulting from weak identification, which often occur in the context of MLE. For instance, Brinca et al. (2018) find that weak identification can be particularly problematic for structural parameters.

Further, we discourage from determining the steady-state values of the different wedges with MLE, instead to compute them from the model's static equilibrium equations in line

[5]Appropriate algorithms for Kalman smoothing are discussed by Hamilton (1994, Chapter 13) or Durbin and Koopman (2012, Chapter 4), among others.

with Lama (2011). In other words, to fix the steady-state values of output and the subaggregates to their nominal average shares of output and the average share of hours worked on the available time budget of the household and, in addition, to estimate the growth rates of the observables to determine the growth rates of all quantities and labor augmenting progress using least square.

Maximum-Likelihood estimation The estimation of DSGE models for a large set of parameters is typically done using Bayesian methods, although MLE involves fewer assumptions. Applying Bayesian estimation is usually meaningful, since the researcher has a structural parametrization in mind and, by association, an idea of probable parameter values. We would like to stress that the application of BCA requires MLE and any restrictions like the Bayesian approaches are questionable. The wedges are superpositions and interactions of a variety of market distortions with an underlying reduced-form stochastic process, which complicates the interpretation of the Markov transitions. Furthermore, Nutahara and Inaba (2012) argue that the VAR(1) strips down a potentially more sophisticated stochastic process. Thus, the estimated parameters are only pseudo-true for the real model. As a consequence, in general, the values of the process' parameters cannot be interpreted, and a priori assumptions of them are meaningless, and even more seriously, may restrict the set of mappable models.

We would also like to point out two technical issues regarding Bayesian methods and BCA. First, to the best of our knowledge, there is no prior that includes all combinations of parameter values that generate eigenvalues of $\mathbf{\Pi}$ less than one and excludes all combinations that do not have these properties. Second, the posteriors of a VAR-driven DSGE model can be multi-modal, which makes the commonly used RWMH algorithms unsuitable.[6]

For a successful MLE, we suggest partitioning the parameters to be estimated into three subsets: The sets $\boldsymbol{\theta}_{\mathbf{\Pi}}$ and $\boldsymbol{\theta}_{\mathbf{\Sigma}}$ that collect the parameters determining $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ as well as the set of structural parameters $\boldsymbol{\theta}_{\mathbf{S}}$ if needed. Further, let $\mathbf{\Sigma} = \mathbf{\Omega}\mathbf{\Omega}^T$ and in turn $\mathbf{\Omega}$ a lower triangular matrix. Note at this point, that these different parameter categories influence the policy function (2) of the model in different ways. First, $\boldsymbol{\theta}_{\mathbf{\Sigma}}$ does not affect the policy function at all, since we use linear approximation techniques. Second, $\boldsymbol{\theta}_{\mathbf{\Pi}}$ only affects the stochastic part of the policy function ($\mathbf{L}_{\mathbf{z}}^{\cdot}$). Last, $\boldsymbol{\theta}_{\mathbf{S}}$ may also affect its deterministic part

[6]For a deeper discussion and solutions for multi-modal posteriors see Herbst and Schorfheide (2015, Chapter 5 and 6.1).

($\mathbf{L}_{\mathbf{x}}^{\cdot}$). The effect on different terms of the policy function of different parameters means, that given a set of structural parameters, we can reevaluate the likelihood function for $\boldsymbol{\theta}_{\Pi}$ and $\boldsymbol{\theta}_{\Sigma}$ without repeatedly solving for the deterministic part of the policy function.

Identification: Most issues with estimating DSGE models are due to a lack of identification. Since this issue is particularly apparent in combination with MLE, it is essential to make sure that the prototype economy is identifiable concerning uncertain parameters. Concerning BCA, Brinca et al. (2018) provide and apply strategies for identification strength. They show that weak identification of the stochastic process' parameters is secondary but primary for structural ones. To address this problem, it is advisable to compute the likelihood surface of uncertain structural parameters to detect a global maximum as well as the likelihood's curvature and to execute robustness checks. Furthermore, we suggest verifying strict local identifiability through Iskrev's 2010 method. Iskrev (2010) shows that a linearized DSGE model with normally distributed shocks is locally identified for a given set of parameters if the Jacobian matrix of the theoretical first and second moments for these parameters has full rank. To check the identifiability over a sufficiently large parameter space, we suggest drawing sufficiently times from a suitable distribution for all uncertain parameters to subsequently verify the identifiability of the model for the drawn parameters.

Likelihood Evaluation: To maximize the likelihood function over the unknown parameters and thus determine the ML estimator of these parameters, the policy function of the log-linearized prototype economy is usually transformed into a state-space model of the form:

$$\mathbf{y}_t = \mathbf{H}\mathbf{w}_t, \tag{7a}$$

$$\mathbf{w}_t = \mathbf{F}\mathbf{w}_{t-1} + \mathbf{v}_t, \qquad \mathbf{v}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{Q}), \tag{7b}$$

with

$$\mathbf{w}_t = \begin{pmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{pmatrix}, \quad \mathbf{v}_t = \begin{pmatrix} \boldsymbol{\epsilon}_t \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \mathbf{L}_{\mathbf{z}}^{\mathbf{y}} & \mathbf{L}_{\mathbf{x}}^{\mathbf{y}} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \boldsymbol{\Pi} & \mathbf{0} \\ \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} & \mathbf{L}_{\mathbf{x}}^{\mathbf{x}} \end{pmatrix}, \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Since the state-space model (7) is linear with Gaussian disturbances $\mathbf{v}_t$ for all $t = 1, 2, \ldots, N$, using the prediction error decomposition we may obtain minus twice the log-likelihood function (without constant) as

$$L = \sum_{t=1}^{N} \ln \left| \mathbf{H}\left(\mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{Q}\right)\mathbf{H}^T \right| + \left(\mathbf{y}_t - \mathbf{H}\mathbf{F}\boldsymbol{\mu}_{t-1}\right)^T \left(\mathbf{H}\left(\mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{Q}\right)\mathbf{H}^T\right)^{-1} \left(\mathbf{y}_t - \mathbf{H}\mathbf{F}\boldsymbol{\mu}_{t-1}\right), \quad (8)$$

where for given $\boldsymbol{\mu}_0$ and $\mathbf{C}_0$ the Kalman filter may be used to compute $\boldsymbol{\mu}_t = \mathbb{E}[\mathbf{w}_t|\mathbf{y}_t, \ldots, \mathbf{y}_1]$ and $\mathbf{C}_t = \mathbb{V}[\mathbf{w}_t|\mathbf{y}_t, \ldots, \mathbf{y}_1]$ for all $t = 1, 2, \ldots, N$ recursively as

$$\boldsymbol{\mu}_t = \mathbf{F}\boldsymbol{\mu}_{t-1} + \left(\mathbf{H}\mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{H}\mathbf{Q}\right)^T \left[\mathbf{H}\left(\mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{Q}\right)\mathbf{H}^T\right]^{-1} \left(\mathbf{y}_t - \mathbf{H}\mathbf{F}\boldsymbol{\mu}_{t-1}\right), \quad (9a)$$

$$\mathbf{C}_t = \mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{Q} - \left(\mathbf{H}\mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{H}\mathbf{Q}\right)^T \left[\mathbf{H}\left(\mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{Q}\right)\mathbf{H}^T\right]^{-1} \left(\mathbf{H}\mathbf{F}\mathbf{C}_{t-1}\mathbf{F}^T + \mathbf{H}\mathbf{Q}\right). \quad (9b)$$

The ML estimate of $\boldsymbol{\theta} = \left(\boldsymbol{\theta}_{\mathbf{S}}^T, \boldsymbol{\theta}_{\Pi}{}^T, \boldsymbol{\theta}_{\Sigma}^T\right)^T$ is then given by

$$\hat{\boldsymbol{\theta}} = \operatorname{argmin}_{\boldsymbol{\theta}} L(\boldsymbol{\theta}) \qquad \text{and} \qquad \hat{\mathbb{V}}\left[\hat{\boldsymbol{\theta}}\right] = \left[-\frac{\partial^2 L\left(\hat{\boldsymbol{\theta}}\right)}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}^T}\right]^{-1}, \quad (10)$$

with approximately $\hat{\boldsymbol{\theta}} \sim \mathrm{N}\left(\boldsymbol{\theta}, \hat{\mathbb{V}}\left[\hat{\boldsymbol{\theta}}\right]\right)$ for large N.

Unconditional Initialization: The most common initialization for a stationary state space model, like the one described by (7), is to specify $\boldsymbol{\mu}_0$ and $\mathbf{C}_0$ as the unconditional mean vector $\boldsymbol{\mu} = \mathbb{E}[\mathbf{w}_t]$ and the unconditional variance matrix $\mathbf{C} = \mathbb{V}[\mathbf{w}_t]$ of state vector $\mathbf{w}_t$ (see e.g., Hamilton (1994, Chapter 13) or Durbin and Koopman (2012, Chapter 5)). Thus $\boldsymbol{\mu}_0$ and $\mathbf{C}_0$ may be obtained as

$$\boldsymbol{\mu}_0 = \mathbf{0}, \quad (11a)$$

$$\operatorname{vec}(\mathbf{C}_0) = (\mathbf{I} - \mathbf{F} \otimes \mathbf{F})^{-1} \operatorname{vec}(\mathbf{Q}). \quad (11b)$$

The major advantage of this initialization strategy is that it provides the exact and unconditional log-likelihood for the state-space model (7). Furthermore, this strategy is consistent with the way we obtain the model's states in (4). However, using initialization (11) usually requires the numerical minimization of (8) with respected to all $n_z^2 + (n_z^2 + n_z)/2$ parameters referring to the stochastic process of $\mathbf{z}_t$. With a decreasing degree of aggregation and thus increasing number of wedges, numerical minimization becomes rapidly infeasible, as with an increasing number of wedges the number of parameters to be estimated increases disproportionately. Even the stochastic process of the maximal aggregated benchmark prototype economy by CKM (four wedges) is determined by 26 parameters.

Conditional Initialization: We propose another initialization strategy that reduces the number of parameters involved in the numerical optimization (and referring to the stochastic process of $\mathbf{z}_t$) to n_z^2. This second initialization strategy is based on the additional assumption that the prototype economy under consideration was in its growth equilibrium at time $t = 0$, i.e., $\mathbf{w}_0 = \mathbf{0}$, and thus leads to

$$\boldsymbol{\mu}_0 = \mathbf{0}, \tag{12a}$$

$$\mathbf{C}_0 = \mathbf{0}. \tag{12b}$$

It follows directly from (9b) that using initialization (12), $\mathbf{C}_t = \mathbf{C}_{t-1} = \mathbf{0}$ must hold for all $t = 1, 2, \ldots, N$.[7] Thus, for the given set $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t$ of data, the states $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t$ of the prototype economy under consideration are fixed (i.e., $\mathbf{w}_t = \boldsymbol{\mu}_t,\ \forall t = 1, 2, \ldots, N$). This is not surprising, since the conditional initialization is based on the same assumption made in (3) to determine the exogenous and endogenous states of the model. This is also the first advantage of the conditional initialization (12) over the unconditional initialization (11). Since $\mathbf{C}_t$ equals zero matrix for all $t = 1, 2, \ldots, N$, the computationally expensive part (9b) of the Kalman filter becomes redundant. However, the biggest advantage of initialization (12) is that the corresponding likelihood function may be concentrated with respect to $\boldsymbol{\theta}_{\boldsymbol{\Sigma}}$. To see this note that we can write (8) conditional on $\mathbf{w}_0 = \mathbf{0}$ as

$$L_C = N \left[\ln \left| \mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\, \boldsymbol{\Sigma}\, \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T \right| + \operatorname{tr}\left(\left(\frac{1}{N} \sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T \right) \boldsymbol{\Sigma}^{-1} \right) \right], \tag{13}$$

where we can use (4) to evaluate $\mathbf{x}_t$ and $\mathbf{z}_t$ for all $t = 1, 2, \ldots, N$.[8] Thus, from differentiating (13) with respect to $\boldsymbol{\Sigma}$ and equating to zero we obtain the (conditional) ML estimator for $\boldsymbol{\Sigma}$ as

$$\operatorname{argmin}_{\boldsymbol{\Sigma}}\ L_C = \hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T. \tag{14}$$

[7] To see this, note that $\mathbf{C}_t = \mathbf{C}_{t-1} = \mathbf{0}$ satisfies (LE.1b), which is due to Lemma E.1 in Appendix E equivalent to the Ricatti equation described by (9b).

[8] A detailed derivation of (13) is given in Appendix E.

Substituting $\hat{\boldsymbol{\Sigma}}$ back into (13) we receive (minus twice) the conditional log-likelihood (without constant) concentrated with respect to $\boldsymbol{\theta}_{\Sigma}$ as

$$L_{C,\hat{\Sigma}} = N\left[\ln\left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\hat{\boldsymbol{\Sigma}}\,\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^{T}\right| + n_z\right]. \tag{15}$$

Thus, using the conditional initialization to estimate $\boldsymbol{\theta}$ is beneficial, since it reduces the dimension of the numerical optimization problem and is therefore often more robust. Further, note that from Proposition E.1 in Appendix E follows, that for any valid initialization $\mathbf{C}_0$ the sequence $\{\mathbf{C}_t\}_{t=0}^{N}$ converges to $\mathbf{C}_* = \mathbf{0}$, if the eigenvalues of the matrix $\mathbf{Z}_2$ are inside the unit circle. Since this condition is usually satisfied in the BCA context,[9] the application of the conditional initialization (12) is equivalent to the procedure of CKM, who initialize their Kalman filter at the long-run equilibrium of the Ricatti equation (9b), i.e., $\mathbf{C}_0 = \mathbf{C}_*$. Moreover, as we show in Appendix E, the constancy of the ML estimator does not depend on the way the filter is initialized as long as the eigenvalues of the matrix $\mathbf{Z}_2$ are inside the unit circle. However, in comparison to the unconditional initialization, the ML estimates based on the conditional initialization are less efficient if the initial states are unknown.

Estimation: To estimate the chosen parameters, we suggest a two-stage estimation procedure: At the first stage, we estimate the parameters of the model based on the conditional ML estimator.[10] In a second estimation we then determine the unconditional ML estimator, using the first-stage estimates as an initial guess for the numerical optimization. Using this procedure we combine the speed and robustness advantages of the conditional ML estimator with the higher efficiency of an unconditional initialization.

2.4 Findings and discussion

Once the wedges states are estimated, the wedges are fed back one by one using the wedge decomposition (5) to evaluate the contribution of each wedge. For the illustration, most exercises plot for each wedge i time series of some of the decomposed control variables $\tilde{\mathbf{y}}_t$ and $\tilde{\mathbf{c}}_t$. Additionally, Brinca et al. (2016) define a summary measure, called ϕ statistic, to

[9] We show in Appendix A that this holds for the prototype economy of our application. In doing so, it also becomes apparent that the opposite should not occur generally.

[10] Note that these estimates combined with the states obtained from (3) are equivalent to the procedure of CKM.

capture the contribution of a wedge to some of the observables $\mathbf{y}_t$. The ϕ statistic reads

$$\phi_i^y = \frac{1/\sum_t (y_t - \tilde{y}_{it})^2}{\sum_{j=1}^{n_z} 1/\sum_t (y_t - \tilde{y}_{jt})^2}, \; y_t \in \mathbf{y_t}, \; \tilde{y}_{it} \in \tilde{\mathbf{y}}_t, \; t = p, p+1, ..., p+k.$$

Note that this measure is the inverse of the mean-squared error for a wedge in comparison to the sum of all wedges' inverse of the mean squared error. Thus, the ϕ statistic is rather suited to assess prediction than contribution as in the case of a countercyclical contribution of any wedge, the statistic can neither indicate this countercyclicality nor differentiate between a potential over- or undershooting behavior of other wedges.[11] Therefore, we propose two other measures, referred as to the Δ statistics:

$$\Delta_i^y = \frac{\sum_t (\tilde{y}_t^{(0)} - \tilde{y}_{it})}{\sum_j \sum_t (\tilde{y}_t^{(0)} - \tilde{y}_{jt})} = \frac{\sum_t (\tilde{y}_t^{(0)} - \tilde{y}_{it})}{\sum_t (\tilde{y}_t^{(0)} - y_t)},$$

$$\bar{\Delta}_i^y = \frac{1}{k} \sum_t \frac{(\tilde{y}_t^{(0)} - \tilde{y}_{it})}{\sum_j (\tilde{y}_t^{(0)} - \tilde{y}_{jt})} = \frac{1}{k} \sum_t \frac{(\tilde{y}_t^{(0)} - \tilde{y}_{it})}{(\tilde{y}_t^{(0)} - y_t)}$$

where $\tilde{y}_t^{(0)}$ is the component of the variable y_t when all wedges are fixed in the basis year $p-1$ ($\tilde{y}_t^{(0)} y_t \in \tilde{\mathbf{y}}_t$ for $\mathbf{S} = \mathbf{0}$). Interpreting $\tilde{y}_t^{(0)}$ as the trend component of a variable, Δ_i^y represents the total contribution of the wedge i to the cyclical component of y. Note that for GDP, this component is the output gap. In the same manner, $\bar{\Delta}_i$ accounts for the average contribution. The Δ statistic is suitable to indicate the total contribution of a particular phase. However, over the cycle, the contribution evens out, so the average contribution $\bar{\Delta}$ must be chosen.

The discussion of the results is against theoretical or economic and political historical equivalent results. Depending on the case, it may be useful to develop different robustness checks and further evaluation methods, for which we give examples in our application.

3 BCA for the Great Recession in Germany

Like in many other countries, the Great Recession hit the German economy massively, and also like many others, in response to the recession, the German government launched an

[11] Overshooting here means that the fluctuation of a variable would be larger than observed if only the overshooting wedge had contributed to it.

expansive fiscal stimulus program. This program was composed of two packages. The first became effective at the end of 2008 and the second at the beginning of 2009 (Bundesgesetzblatt, 2008, 2009).

As Rosenberger (2013) describes, the first package amounted to 32 Billion € plus a loan program of 15 Billion €. The fiscal stimulus consisted of a one-year tax exemption on new cars, higher tax deductions by permitting the reducing-balance method and increasing child allowance, a lower employment insurance tax, as well as higher transfers for students and retirees.

The second stimulus package amounted to 50 Billion € plus both a loan and guarantee program of 100 Billion € and an increase of the German export credit guarantee program (Hermes cover) of about 2 Billion €. The package consisted of investments in public infrastructure, financial support for local and state authority spending, a subsidy on new cars at the amount of 2500 € per car and in total 5 Billion €, subsidies for private innovations as well as lower income taxes and social contributions. Short-time work possibilities and benefits were expanded, further training was supported, and the Federal Employment Agency increased the number of job agents.

Table 1 presents calculations by the OECD (2009) for the stimulus program. The size of the fiscal stimulus program was on equal terms by reducing taxes and increasing transfers and spending. Transfers to households amounted to 0.3% of GDP, where the cash for clunkers composed two out of three. Extra government spending amounted to 0.8% of GDP. The fiscal packages amounted to 3.2% of GDP, excluding all measures which did not affect the national budget directly, e.g., the loan and guarantee program.

Additionally, the monetary policy reacted to the Great Recession. The European Central Bank (ECB) lowered the minimum bid rate on main refinancing operations and the interest rate on deposit facilities. Additionally, the ECB applied unconventional tools of monetary policy by doing quantitative easing.

Table 1: Composition of the fiscal program in % of GDP

Tax	Individuals	Social Contribution	Business	Total*
	-0.6	-0.7	-0.3	-1.6
Spending	Transfers to households	Transfers to business	Government spending**	Total***
	0.3	0.3	0.8	1.6

Notes: * Including consumption tax measures. ** Final consumption + investment *** Including transfers to sub-national government. **Source:** OECD (2009).

3.1 Data processing

First note briefly, we take the data from the Fachserie 18 from the German Federal Statistical Office. Appendix C lists the data source in detail.

Aggregation For the aggregation level, we take the fiscal stimulus program into account. Additionally, we consider that the German economy is heavily dependent on exports and that there was a sharp drop in the demand for export goods since it was a global crisis. Thus, we conclude to separate government consumption and net exports as they shape the behavior and expectations of the economy's agents in different ways. Interpreting the cash for clunkers program and the tax exemption for new cars as subsidies for durable consumption, we need to distinguish between durable consumption goods and other investment goods according to Greenwood and Hercowitz (1991).[12] All in all, the demand is composed of the following individual goods: Consumption, durable consumption, investment, net exports, and government consumption.

Testing trends The choice of observables falls on GDP, durable consumption, investment, net exports, government consumption, and hours worked. The balanced growth hypothesis implies that the former five quantities share the same growth factor. We test for a common trend of these quantities. Columns 2 and 3 of Table 2 presents the results whether the quantities share the average GDP growth factor and columns 4 and 5 test for a common trend growth factor in the manner of DeJong and Dave (2011, Ch. 6.1). The latter extends ordinary least squares estimation by the linear restriction that all growth rates are equal. The balanced growth hypothesis is rejected on a 1 percent significance

[12] Residential investment is not separated from business investment since the homeownership rate in Germany is not above 50% and no significant subsidy or tax changes have taken place.

level for both growth factors. Thus, we detrend all quantities with their average growth rate. Further, we test the stationarity of hours worked in columns 7 and 8. We cannot reject the hypothesis of no trend growth in the amount of labor. Thus, we do not detrend hours worked.

Table 2: Testing for trends

Test	GDP trend		DD trend		CV	Labor trend		CV
	No PW	VAR(1) PW	No PW	VAR(1) PW	$p < 0.01$	No PW	VAR(1) PW	p<0.1
VF1	2479.2	2666.2	2966.2	1341.8	78.3	0.064	0.040	4.56
VF2	1638.6	1036.6	1963.9	1309.6	70.1	0.110	0.072	5.22
HAC	1579.4	445.0	1488.0	1430.5	15.1	0.064	0.130	1.64

Notes: GDP and DD trend test $H_0: \; g_{GDP} = g_G = G_I = G_D = G_{NE} = g^*$, where g^* is the average GDP growth factor and the common growth factor DeJong and Dave (2011, Ch. 6.1), respectively, where g_{GDP} is the GDP, g_G the government consumption, g_I the investment, g_D the durable, and g_{NE} the net exports trend growth factor. Labor trend tests $H_0: \; g_N = 0$ with g_N the trend growth factor of hours worked. Test statistics: VF1: Vogelsang and Franses (2005) Test 1. VF1: Vogelsang and Franses (2005) Test 2. HAC: Newey-West heteroskedasticity and autocorrelation consistent. CV: critical values. PW: prewhitening.

Exploration Table 3 presents in the second column the average nominal shares of sub-aggregates of the reunified German economy (1991–2018). Private consumption expenditures (PCE) account for 57%, whereby durables account for 6% and non-durables for half of GDP. The share of investment is at 20% and of government consumption close to 19%. Net exports account for almost 4%. The average share of hours worked in the available time budget of a household is more than 12%.[13]

[13] Here we follow (Heer and Maussner, 2009, Chapter 1.5), who assume that the household's time endowment amount to $1,440 = 16$ hours per day$\times$90 days per quarter.

Table 3: Average ratios in % of GDP (1991–2018)

x	$\overline{p_{xt}x_t/\text{GDP}_t}$ [a)]	g_x
GDP	100.0%	1.32%
Non-durables consumption	50.37%	1.03%
Durable consumption	06.41%	0.35%
Investment	20.36%	0.93%
Government consumption.	18.72%	1.40%
Net exports	03.76%	1.65%
Hours worked	12.19% [b)]	0.00%
Population	–	0.03%

[a)] Average share of the nominal value of x on nominal GDP, g_x: annual growth rate of x.

[b)] Share of hours worked in the total time budget.

Source: See Appendix C, own calculations.

The third column of Table 3 presents the growth rates of GDP, the real subaggregates, hours worked, and the population. The GDP annual trend growth rate is 1.32%. The amount of non-durable and durable consumption and investment goods grows slower than GDP, while net exports grow faster. Government consumption grows similar to GDP. Differences in the growth rates of the durables and investment goods may occur due to investment-specific technological change as described by Greenwood et al. (1997). The increase in German net exports since the launch of the Euro is investigated by in't Veld et al. (2014). The most important factors are a higher German savings rate, positive supply shocks, especially due to labor market reforms, as well as higher demand for German goods of non-Euro area members.[14]

Figure 1 presents the cyclical behavior of GDP. We observe a boom-bust cycle in GDP at the same time as the dot-com bubble. This cycle was followed by a recovery from 2005 till 2008, which ended in a heavy drop. This drop depicts the Great Recession. GDP recovered fast and has moved along the long-run trend since then.

Figure 2 presents the cyclical behavior of the subaggregates of demand and hours worked.

[14]The discussion of different growth rates could be interpreted as the growth accounting counterpart to business cycle accounting. Thus, it is a matter of taste whether to discuss this here or in the results section.

Panel (a) shows that investment has co-moved with GDP, but with higher volatility. Panel (b) displays two short but heavy boom-bust-cycles of durables. The first peaked at the end of 2006, shortly after the announcement of a value-added tax (VAT) increase. This was followed by a bust at the beginning of 2007 when the increase took place. We observe the second peak at the same time as the German cash for clunkers program, which was also followed by a bust as the program expired.

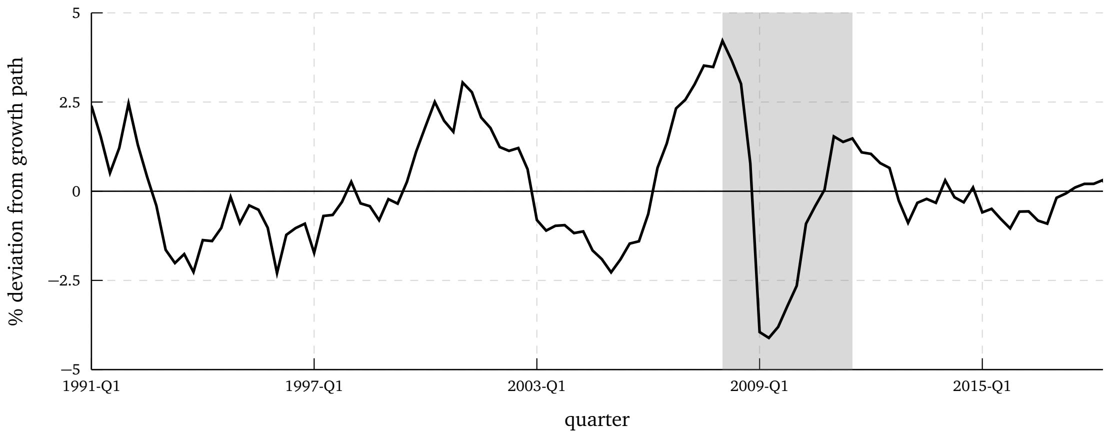
Government consumption was above its trend in the middle and late 1990s. It decreased at the beginning of the 2000s and increased from 2008 till 2010. Since 2010, it has fluctuated around its trend. Non-durable consumption was below its trend in the aftermath of the reunification, and was above the trend in the 2000s until the Great Recession and decreased slightly afterward. Net exports relative to GDP decreased from 1997 till 2001 from their trend and increased sharply afterward till 2003. From then on until the crisis they moved above the trend. Since the crisis, they have fluctuated around the trend. In the medium-run, hours worked declined after the German reunification till 2005 and from then on they have increased. Hours worked have co-moved with GDP from 2000 onwards.

The gray area in Figures 1 and 2 indicates the Great Recession. GDP, hours worked and investment decreased from the end of 2008 until the peak of the crisis in 2009-Q2 by 5, 4, and 12 percentage points, respectively. Their recovery was completed in 2011. Durables increased during the time of the car subsidies by 12 percentage points and decreased by 18 percentage points afterward. Durables recovered at the end of 2010. Government consumption increased at the beginning of 2009 by 5 percentage points and remained till the end of 2011 by 4 percentage points above its trend. Non-durables were less than 2% below their trend at the end of 2009 and recovered fast.

3.2 The prototype economy

We extend the model of CKM in three ways to be in line with our empirical work above. Firstly, wedges consist of a growth and a business cycle part to separate growth and business cycle accounting and ensure stationarity of the stochastic process and secondly, we distinguish between government spending and net exports and exclude durables from aggregated investment goods. Lastly, the model also accounts for productive capital and durable consumption capital adjustment costs. Chang (2000) shows that adjustment costs for capital goods in the market and at home solves problems with excess volatility and negative co-movements because adjustment costs lower the substitutability, which is why

Figure 1: Cyclical behavior of GDP



Notes: The data is presented as relative deviations from linear trend. The gray area indicates the crisis and the recovery from 2008-Q1 – 2011-Q3. **Source:** see Appendix C, own calculations.

we model this structural friction explicitly. Interpreting adjustment costs as a restriction to the technical transformation, it is not a deviation from the neoclassical perspective and thus not a deviation from the interpretation of the wedges.[15] The model is written in per capita terms.

3.3 Model

The per period utility of the representative household is parameterized as follows

$$u(C_t, D_t, N_t) = \begin{cases} \phi \ln(C_t) + (1-\phi)\ln(K_{Dt}) + \psi \ln(1-N_t) & \text{for } \eta = 1, \\ \frac{\left(C_t^{\phi} \cdot K_{Dt}^{1-\phi} \cdot (1-N_t)^{\psi}\right)^{1-\eta} - 1}{1-\eta} & \text{for } \eta \neq 1, \end{cases} \tag{16}$$

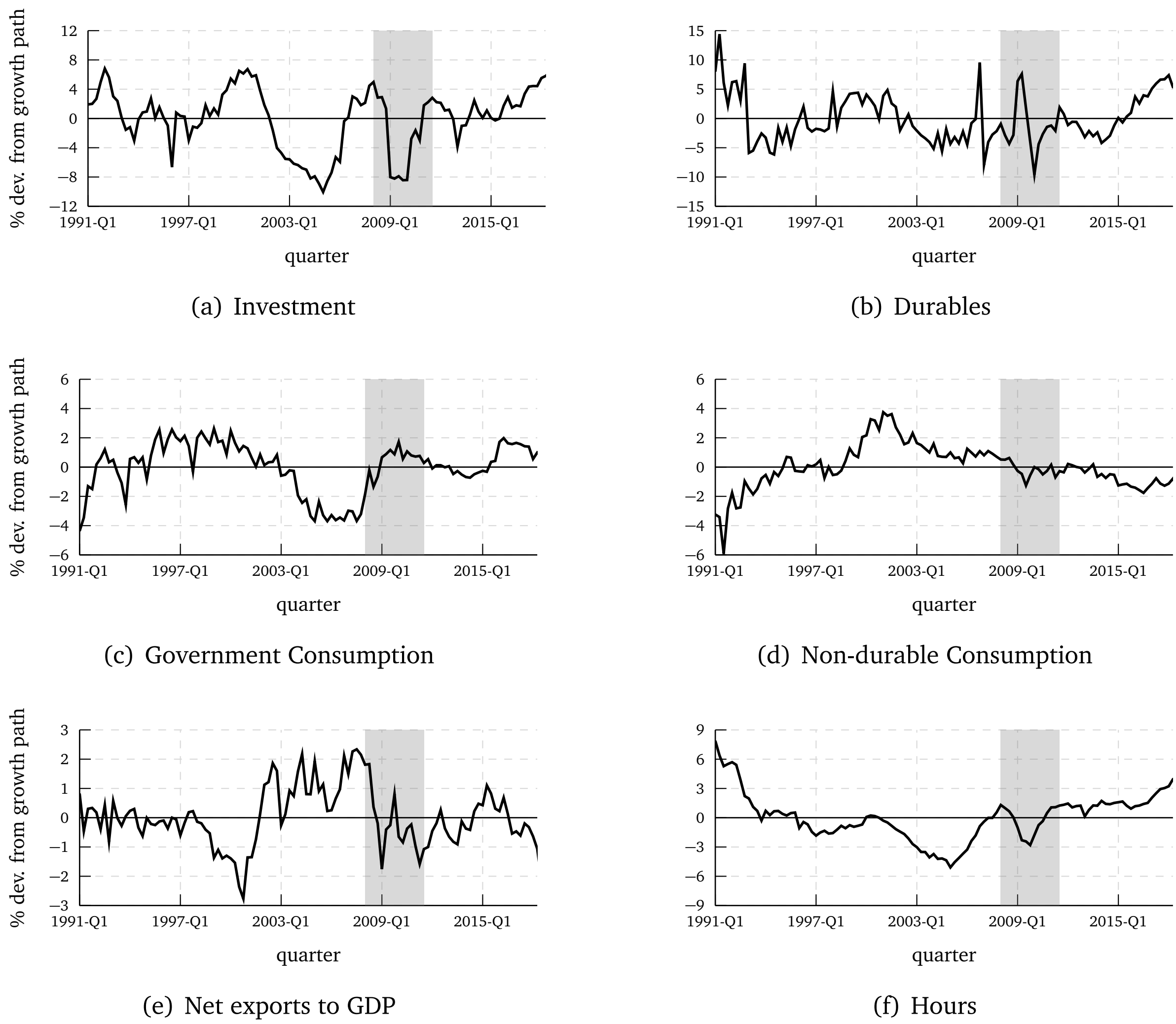
where C_t denotes consumption of non-durable goods and N_t is the household's labor supply. The stock of durable consumption goods K_{Dt} accumulates according to

$$\gamma_n K_{Dt+1} = (1-\delta_D)K_{Dt} + D_t - \Theta_{Dt}\left(\frac{D_t}{K_{Dt}}\right)K_{Dt}, \; \Theta_{Dt}\left(\frac{D_t}{K_{Dt}}\right) = \frac{a_D}{2}\left(\frac{D_t}{K_{Dt}} - b_D\right)^2, \tag{17}$$

where γ_n denotes the population growth factor, D_t are investments in durable consumption goods, and b_D is the ratio of investment in durables to the stock of durables in the long

[15] We give an overview on the wedges' interpretation and their mapping toward the fiscal stimulus program in Appendix A.

Figure 2: Cyclical behavior of different economic measures



Notes: Except hours worked, the data are presented as relative deviations from the corresponding linear trend. Hours worked is the percentage deviation from the average. The gray area indicates the crisis and the recovery from 2008-Q1 – 2011-Q3. **Source:** see Appendix C, own calculations.

run. The household maximizes its expected life-time-utility

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\gamma_n)^s u(C_{t+s}, K_{Dt+s}, N_{t+s}) \tag{18}$$

with respect to the stream of consumption, durables and labor and subject to the budget constraint

$$C_t + (1+\tau_{It})P_{It}I_t + (1+\tau_{Dt})P_{Dt}D_t \leq R_t K_{It} + (1-\tau_{Nt})W_t N_t + T_t - P_{Et}E_t, \tag{19}$$

where K_{It} denotes the productive capital stock (capital stock hereafter), I_t investment in capital, T_t lump-sum transfers, E_t net exports, R_t the rental rate on capital, and W_t the real wage. The tax rates τ_{Nt}, τ_{It} and τ_{Dt} are used to model wedges in the labor, investment and durables market. P_{Et}, P_{It} and P_{Dt} are the relative prices for net exports, investment, and durable goods. The consumption good is the numeraire. The capital stock follows the law-of-motion

$$\gamma_n K_{It+1} = (1-\delta_I)K_{It} + I_t - \Theta_{It}\left(\frac{I_t}{K_{It}}\right)K_{It},\ \Theta_{It}\left(\frac{I_t}{K_{It}}\right) = \frac{a_I}{2}\left(\frac{I_t}{K_{It}} - b_I\right)^2, \tag{20}$$

with b_I as the investment-to-capital ratio in the long run.

The representative firm faces perfect competition, produces output Y_t, which can be used for (government) consumption, new durables, or investment. The firm maximizes its profits

$$Y_t - W_t N_t - R_t K_t \tag{21}$$

with respect to capital and labor and subject to the output's Y_t production function

$$Y_t = K_{It}^{\alpha}(\gamma_a^t A_t N_t)^{1-\alpha}. \tag{22}$$

The parameter γ_a denotes the growth factor of labor augmenting technical progress and A_t the efficiency wedge.

The government expenditures in consumption units $P_{Gt}G_t$ are exogenous and the government chooses lump-sum transfers T_t, so that its budget constraint

$$P_{Gt}G_t + T_t \leq \tau_{Nt}W_tN_t + \tau_{It}P_{It}I_t + \tau_{Dt}P_{Dt}D_t \tag{23}$$

always binds. Thereby, the resource constraint of the economy is

$$Y_t = C_t + P_{It}I_t + P_{Dt}D_t + P_{Gt}G_t + P_{Et}E_t. \tag{24}$$

Growth component In the long run $P_{Xt} \in \{P_{It}, P_{Dt}, P_{Gt}, P_{Et}\}$ evolves with $P_{Xt} = g_{P_X} P_{Xt-1}$, which determines the long-run component of the wedges. The ensuing trend growth factors of different variables X_t are described in Table 4. These variables are scaled by $x_t = \frac{X_t}{g_X^t}$ and are thus stationary variables. Since the cyclical component includes the steady-state component, detrended prices $p_{Et},\ p_{Gt},\ p_{It},\ p_{Dt}$ are normed to one.

Table 4: Growth factors

X_t	Y_t	C_t	W_t	T_t	I_t	K_{It}	R_t	D_t	K_{Dt}	G_t	E_t	γ_a	N_t	P_{Xt}
g_X	g_Y	g_Y	g_Y	g_Y	g_I	g_I	g_Y/g_I	g_D	g_D	g_G	g_E	$g_Y^{\frac{1}{1-\alpha}} g_I^{\frac{\alpha}{\alpha-1}}$	1	$g_{P_X} = \frac{g_Y}{g_X}$

Business cycle component The VAR(1)-process

$$\mathbf{z}_{t+1} = \mathbf{\Pi}\,\mathbf{z}_t + \boldsymbol{\epsilon}_{t+1},\ \boldsymbol{\epsilon}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{\Sigma}), \tag{25}$$

drives the fluctuation of the model, where $\mathbf{z}_t = \begin{pmatrix} \ln(z_{At}) & z_{Nt} & z_{It} & z_{Dt} & z_{Et} & \ln(z_{Gt}) \end{pmatrix}^T$ and $\boldsymbol{\epsilon}_t = \begin{pmatrix} \epsilon_{At} & \epsilon_{Nt} & \epsilon_{It} & \epsilon_{Dt} & \epsilon_{Et} & \epsilon_{Gt} \end{pmatrix}^T$ and $\mathbf{\Sigma}$ is positive definite. The stochastic process affects the wedges as follows

$$A_t = A^* \cdot z_{At}, \qquad \tau_{Nt} = \tau_N^* + z_{Nt}, \qquad \tau_{It} = \tau_I^* + z_{It},$$
$$\tau_{Dt} = \tau_D^* + z_{Dt}, \qquad e_t = e^* + z_{Et}, \qquad g_t = g^* \cdot z_{Gt},$$

where $A^*,\ \tau_N^*,\ \tau_I^*,\ \tau_D^*,\ e^*$ and g^* are the corresponding steady-state component of the different distortions. Similar to CKM, we define the six wedges as follows: The efficiency wedge A_t, the net export wedge e_t, the government spending wedge g_t, the labor wedge $1-\tau_{Nt}$, the investment wedge $\frac{1}{1+\tau_{It}}$, and the durables wedge $\frac{1}{1+\tau_{Dt}}$. The latter two are defined so that, similar to the labor market wedge, increases act like subsidies and decreases act like taxes in comparison to the steady-state value.

We present in Appendix A the full dynamic equilibrium of the model in stationary variables and the mapping of the policy measures towards the wedges. Using $\hat{x}_t = \ln(x_t) - \ln(x^*)$ as the approximation of the relative deviation of a variable x_t from its steady state value x^*, the policy function of the log-linearized and solved version of our prototype economy may be described by (2) with $\mathbf{x}_t = \begin{pmatrix} \hat{k}_{It} & \hat{k}_{Dt} \end{pmatrix}^T$ and $\mathbf{y}_t = \begin{pmatrix} \hat{y}_t & \hat{N}_t & \hat{i}_t & \hat{d}_t & \hat{g}_t & \widehat{\frac{e_t}{y_t}} \end{pmatrix}$.

3.4 Quantification

Calibration We estimate the elasticity, $\eta_I = \frac{I}{K_I}\Phi_I''$, of the price of capital with respect to the investment to capital ratio as well as the elasticity, $\eta_D = \frac{D}{K_D}\Phi_D''$, of the price of the stock of durables with respect to the new durables to stock of durables ratio in addition to the parameters that characterize the stochastic process $\mathbf{z_t}$. The remaining parameters are calibrated as follows:

The capital elasticity α is set to 0.34. Flor (2014) calculates this as the German capital share from 1991 to 2012. In line with Heer and Maussner (2009, Chapter 1.5), Flor (2014) also provides the discount parameter $\beta = 0.994$ for the German economy. We pin down the annual rate of capital depreciation at the average ratio of gross fixed capital formation and the net stock of fixed assets. The average quarterly capital depreciation rate arises from $\delta_I = 1 - (1 - \delta_{I,annual})^{\frac{1}{4}}$. In the same manner, the rate of durables depreciation δ_D is computed.

The choice of ψ, ϕ, and η, which characterize the household's preferences, is more problematic. For ψ and η, we follow the baseline calibration from CKM and fix ψ at 2.24 and η at 1. We calibrate the preference weight of durables ϕ by matching the durable to non-durable consumption ratio with the long-run marginal rate of substitution between consumption and durables. We fix the steady-state values of output, government consumption, investment in capital as well as in durables to their nominal average shares of output (see Table 3). The steady-state labor supply N is 0.122, which equals the average share of hours worked on the available time budget of a household.

Table 5: Calibration of the model

Parameter	Description	Value
α	Capital share	0.34
β	Discount factor	0.994
δ_I	Rate of capital depreciation	0.017
δ_D	Rate of durables depreciation	0.045
ψ	Preference weight of labor	2.24
ϕ	Preference weight of consumption	0.879
η	Risk aversion	1

Identification We check our prototype economy for strict local identification following Iskrev (2010). To check the identifiability over a sufficiently large parameter space we draw 1,000,000 times from the following distributions for the elasticities of the adjustment costs η_D, η_I, for the the off-diagonals π_{ij}, $i \neq j$ of $\mathbf{\Pi}$, for the diagonals π_{ii} of $\mathbf{\Pi}$, and the elements ω_{ij}, $i \leq j$ of the lower triangular matrix $\mathbf{\Omega}$ with $\mathbf{\Sigma} = \mathbf{\Omega}\mathbf{\Omega}^T$:

$$\eta_D, \eta_I \sim \mathrm{U}(0,4), \qquad \pi_{ij} \sim \mathrm{N}(0,0.1), \qquad \pi_{ii} \sim \mathrm{N}(0.8,0.1), \qquad \omega_{ij} \sim \mathrm{U}(-0.05,0.05).$$

The Jacobian of the first and second moments (up to two lags) has full rank at approximately 99.9 percent of the draws. Thus, the model is virtually identifiable in the chosen parameter space.[16]

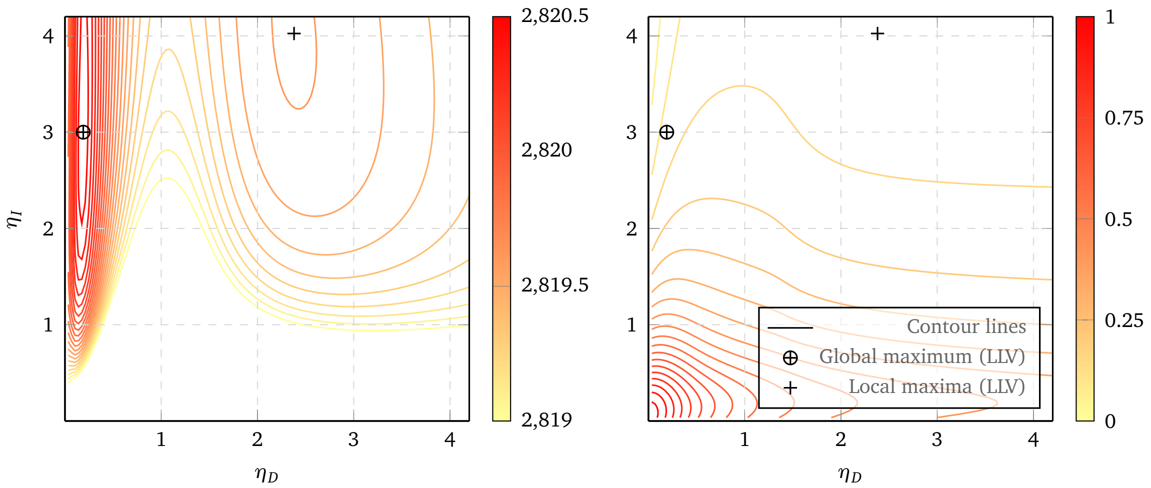
Estimation We use our procedure to estimate $\mathbf{\Pi}$, $\mathbf{\Sigma}$, η_D and η_I. Panel 3(a) illustrates the likelihood function with respect to η_D and η_I, while Π and Σ are the argument maximum of the function for given η_I and η_D. The panel identifies two local maxima. The global is at $\eta_D = 0.19$ and $\eta_I = 3.00$. Table A.1 in Appendix A presents the estimates for the autoregressive matrix $\mathbf{\Pi}$ and the innovations ϵ_i.

Panel 3(b) illustrates that the innovations of durables and investments are perfectly correlated in the absence of adjustment costs. Fehrle (2019) investigates different investment goods, vector-autoregressive processes, and adjustment costs in detail and argues that adjustment costs can be viewed as an underpinning mechanism of reduced-form correlated

[16] In comparison, we proceed similarly for the benchmark economy of CKM presented in Appendix B. The Jacobian of the first and second moments (up to two lags) has no full rank at 26 parameter draws from 1,000,000.

shocks. Here, e.g., the mentioned high substitutability between durables and investments is prevented either by perfect correlated innovations, adjustment costs, or a combination of them. Hence, it is useless to separate investments and durables without adjustment costs, since the corresponding wedges must co-move. Otherwise, the high substitutability would lead to excessive volatility of durables and investments and negative co-movements between them. However, this is contradicted by the data.

Figure 3: Maximum-Likelihood-Estimation



(a) Log-Likelihood (without constant) (b) Correlation between ϵ_{It} and ϵ_{Dt}

3.5 Evaluation, discussion and robustness of Business Cycle Accounting results

Evaluation Figure 4 illustrates the graphical analysis of our BCA exercise. In Panels 4(a) to 4(e), we confront the observations of GDP, its (endogenous) subaggregates of demand and hours worked during 2008-Q1,...,2011-Q3 with the model's prediction when only one wedge is allowed to fluctuate.

Panel 4(a) indicates that the crisis was mainly driven by the efficiency wedge. The investment and net exports wedge also contributed to the crisis. These three wedges together induced the decrease in GDP. The labor wedge contributed to the crisis from 2009-Q2 to 2009-Q4. Before, the wedge was counter-cyclical and afterward, it introduced the recovery. The durables wedge and government consumption were anti-cyclical. Panel 4(b) illustrates that the investment wedge drove the decline in investment mostly, while the efficiency wedge mattered little. The efficiency wedge influenced durables negatively as

Panel 4(c) shows. The durables wedge increased durables up to almost 50% in 2009. Afterward, the wedge only had a slight impact. Panel 4(d) indicates that the efficiency wedge caused the decline in non-durable consumption mostly and the labor wedge partly. The durables and government consumption wedge had little impact on non-durable consumption. Panel 4(e) predicts the decline in net exports to GDP and the investment wedge introduced the decline in hours worked. The labor market wedge drove the decline between 2009-Q2 and 2009-Q4. Besides, the labor wedge was counter-cyclical. The other wedges were counter-cyclical.

The measurement Δ_i for $t \in [2008\text{-}Q1, ..., 2011\text{-}Q3]$ quantifies the absolute contribution of each wedge to GDP during the Great Recession. The efficiency wedge accounts for 62% of the decline in GDP during the Great Recession in Germany, net exports for 26%, the investment wedge for 19%, and the labor market accounts for 3%. Government consumption accounts for -5% and the durables wedge for -4%.

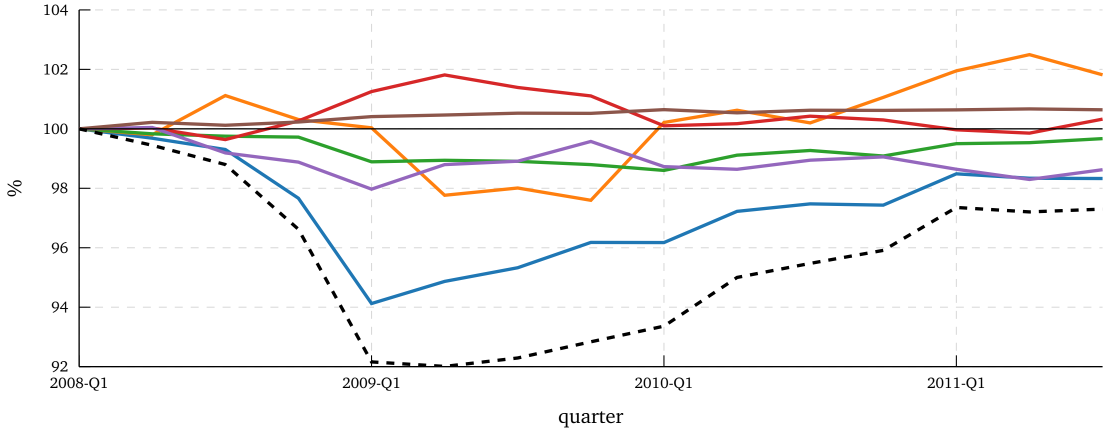
Discussion First, the labor market wedge-induced recovery can be explained by expanded short-time work possibilities as they can decrease hiring frictions in the aftermath of recessions. Using the unemployment rate, Gehrke et al. (2019) argue that previous labor market reforms (so-called Hartz reforms) probably drove the labor market wedge-induced recovery. Unfortunately, BCA cannot distinguish between these explanations because both achieve equivalent results.

Next, we try to gain deeper insights into the drivers of the durable wedge by the process of elimination and comparisons. Subsequently, the results are reviewed against the literature.

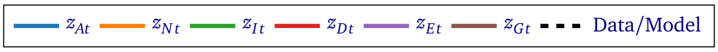
Theory teaches us that the wedges of both investment goods D_t and I_t generally react similarly to frictions, like financial ones.[17] Thus, as discussed, CKM and many others aggregate them. Our results show that the business investment wedge drove the decline in business investment during the crisis. Financial frictions and other distortions – equivalent to a decreasing investment wedge – diminished investment activities. This is empirically not observable for durables. The only appreciable difference between the wedges during the crises was the car subsidies. Further, the positive impact of the durables wedge

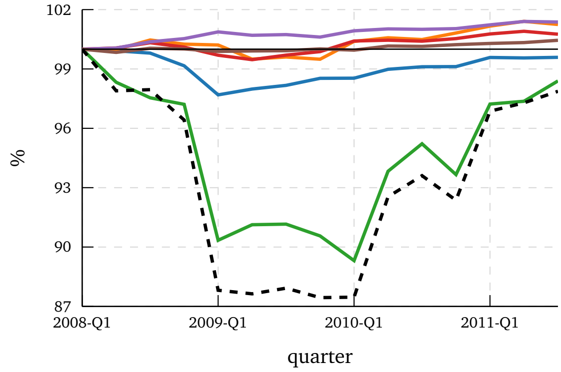
[17] Gertler and Gilchrist (2018) report for the U.S., financial frictions during the Great Recession induced a big negative impact on the durables market. E.g., Benmelech et al. (2017) explain one-third of the decline in the U.S. car demand by frictions on the asset-backed commercial paper market. The decline in U.S. house prices weakens the household balance sheets, which also harmed the U.S. auto market, as shown by Mian et al. (2013).

Figure 4: BCA - Results

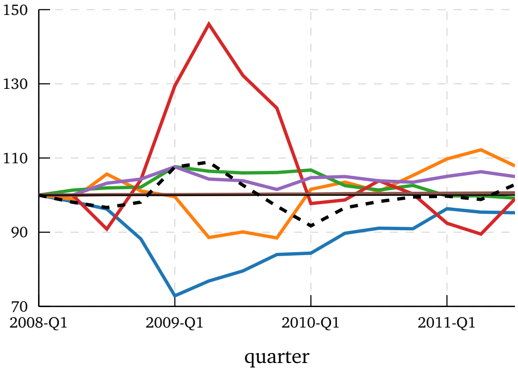


(a) GDP

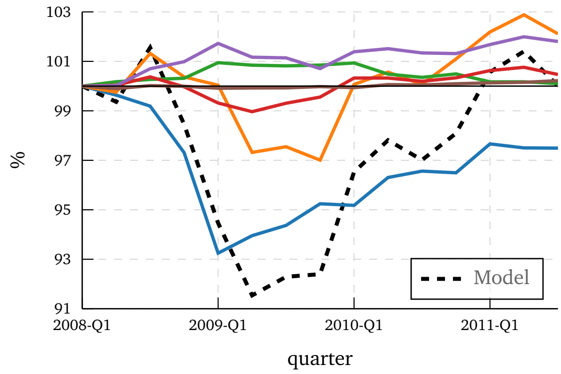




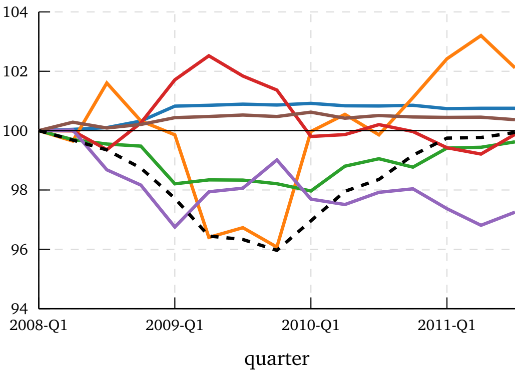
(b) Investment



(c) Durables



(d) Non-durable Consumption



(e) Hours

Notes: Dashed lines for GDP, investment, durables and hours are the data and the model's outcome. Here they are equivalent. The dashed lines for non-durable consumption is only the model's outcome.

occurred simultaneously with the subsidies. The wedge began to stimulate the demand for durable goods with the introduction of the tax exemption for new cars in 2008-Q4. In 2009-Q1 the cash for clunkers program started, while the stimulating effect increased strongly. The stimulus disappeared between 2009-Q4 and 2010-Q1 while the last pay-off took place in 2009-Q4. Hence, mapping the large increase due to the durables wedge to the car subsidies is reasonable. Given that assumption, the cash for clunkers program would have had a sizable effect on aggregate demand and, at least prompt, intertemporal substitution of durables investment in the aftermath of the program was small. The bust was driven by the efficiency wedge, which depressed the quantity of durables over the whole period.

Mian and Sufi (2012) examine a similar cash for clunkers program in the U.S. They find that the program induced a large increase in car sales. However, in their study, the positive effect vanishes within one year due to intertemporal substitution. In Germany, durable goods bust after the program, which suggests a similar substitution effect. But as mentioned, our BCA analysis indicates that this bust is the transmission towards the trajectory of durables that would have occurred in the absence of the cash for clunkers program. In contrast, Leuwer and Süssmuth (2018) find large intratemporal substitution effects for the German cash for clunkers program. However, their work relies on the strong assumption that there were no substantial changes simultaneously to the car subsidy.

There was a second boom-bust cycle in the durable market during a VAT increase announcement and its implementation (2006-Q1 to 2007-Q4). As this policy comes with the same intertemporal substitution effect as the cash for clunkers program, we compare both policies in Figure 5. Note that the income effects are different. Panels 5(a) and 5(b) show the data and the impact of the durables wedge on durables from 2008-Q1 to 2010-Q4 and from 2006-Q1 to 2007-Q4. The durables wedge accounts during the car subsidies programs for the boom, but only marginally for the bust afterward. During 2006, the VAT increase announcement passed the institutions, and during this time durables investments increased. The VAT increase was enforced in 2007-Q1 when the bust took place. The durables wedge caused the whole boom-bust cycle and illustrates prompt intratemporal substitution.

Berger and Vavra (2015) generally investigate the households' responses to durables subsidies over the business cycle for the U.S. and find smaller effects during recessions. Buettner and Madzharova (2020) find high intertemporal substitution effects of durables in the European Union identified by VAT change announcements. The German govern-

Figure 5: The durables boom-bust cycles 2008-2010 and 2006-2007 in comparison

(a) Cash for clunkers program (b) VAT increase with announcement

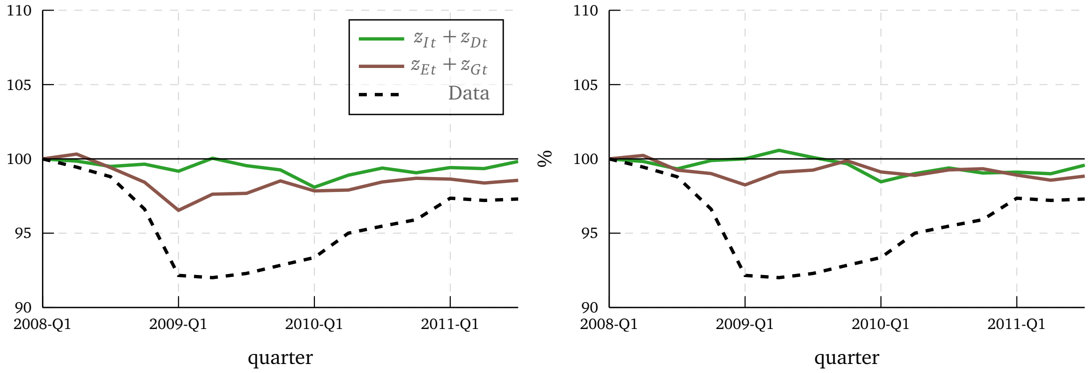
ment adopted a temporary reduction of the VAT in the second half of 2020 to stimulate demand. This reduction has the same intertemporal substitution effect as the policies under investigation and comes with a positive income effect. Clemens and Röger (2021) and Bachmann et al. (2021) find sizable effects on consumption especially on durables with a multiplier larger than one on GDP due to this reduction.

Insights from disaggregation: The left panel of Figure 6 plots the impact of the investment and government spending wedge in the CKM benchmark economy, where durables and investment as well as government spending and net exports are aggregated ex ante.[18] Although the impact of the composed investment wedge was negligible during the Great Recession, our results suggest that the decomposed wedges were not. The pro-cyclical effect of the investment wedge and the policy-driven counter-cyclical effect of the durable's wedge offset each other. Hence, without our decomposition the importance of the investment wedge and, by association, the importance of financial frictions during the Great Recession is underrated. For example, the financial frictions of Carlstrom and Fuerst (1997), Kiyotaki and Moore (1997), Bernanke et al. (1999), or Gertler and Kiyotaki (2010) are equivalent to the investment wedge. The same as for the aggregation of durables and other investment goods holds for the aggregation of government consumption and net exports.

The assessment of the joint contribution of the investment and durables wedge as well as the joint contribution of government consumption and net exports maps our economy into the benchmark BCA economy ex post. The right panel of Figure 6 illustrates these

[18] Appendix B sketches the model and provides our estimation strategy and results for the CKM benchmark economy of the presented time series.

Figure 6: Robustness to the CKM benchmark economy



(a) CKM - Benchmark Economy

(b) Detailed Economy

effects. The results are similar, except in the more detailed economy the investment wedge was slightly counter-cyclical during the cash for clunkers program. Thus, the results of the detailed model are not different from the benchmark BCA model but provide deeper insights.

Robustness in parameters: The presented results depend potentially on the values of adjustment costs η_I, η_D, and on the intertemporal elasticity of substitution η. To evaluate the sensitivity, we calculate Δ_i^{GDP} over a grid of the mentioned parameters. Therefore, we reestimate the (remaining) uncertain parameters at each node of the parameter grid.

Figure 7 illustrates the contribution of the concerning wedges for different amounts of adjustment costs. The efficiency wedge contributed the most to the decline in GDP, followed by net export for the whole set of adjustment costs. The results for the labor market wedge and government consumption are robust as well. The durables wedge mitigated the crisis for most of the parameter combinations. The contribution would have been pro-cyclical without adjustment costs. However as mentioned above, in the absence of adjustment costs a separation of the durables and investment wedge is meaningless. The investment wedge's contribution to the crisis would have been negative for $\eta_I < 1/3$ where the likelihood is the lowest (see Panel 3(a)) and positive otherwise.

Subsidies in durables change the intertemporal rate of substitution. Hence, a robustness check to the elasticity of the substitution rate is relevant. Figure 8 presents the contribution to the decline in GDP over η. The contributions of the labor, investment, durables, and government consumption wedge are nearly constant. The contribution of net exports declines with higher elasticity, nevertheless they contributed the second most over the whole domain. The contribution of the efficiency wedge increases with η.

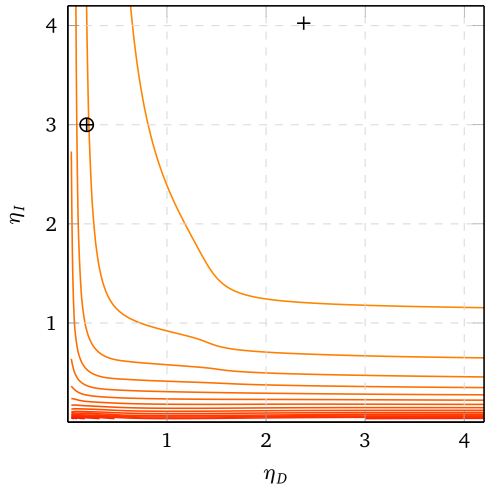
4 Conclusion

Business Cycle Accounting (BCA) is a powerful tool to detect the origins and forces of a particular period of economic fluctuations or general business cycle patterns. Therefore, a multiplicity of applications exists for good reasons, yet, this multiplicity comes with a lack of a precise, standardized procedure to prevent flawed implementation and guarantee comparability. The purpose of the present paper is to bridge this gap by honing the neoclassical lens. To this end, we first author a manual that guides through, refines, and clarifies all the basic steps of an accurate implementation. Subsequently, we illustrate the necessity and benefits of all points raised in an application of BCA toward the Great Recession in Germany and the related stabilizing policy measures.

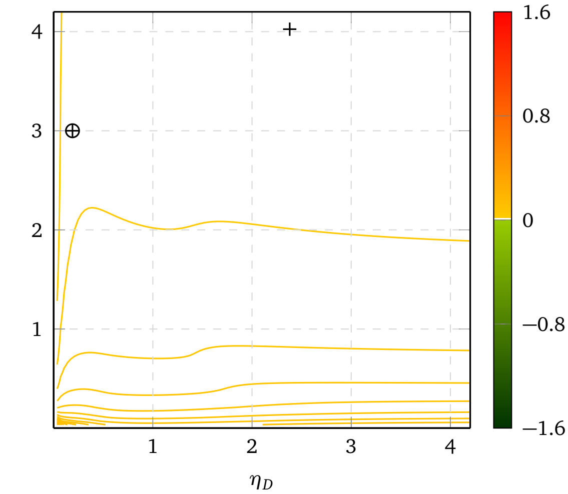
In the manual, we first discuss data preparation. There, the sensibleness of the standard aggregation must be verified as otherwise severe distortions can remain undetected. Concerning trend removing, it is necessary to be i) consistent with the model and ii) guarantee stationarity in all time series. Linear detrending is consistent with the model and, if the balanced growth hypothesis is untenable, one trend per time series guarantees stationarity. In the modeling step, we advise how to disaggregate the quantities in the prototype economy and how to implement different trend growth rates for quantities if necessary. Subsequently, we guide through the numerical implementation of the exercise, where we present our quick and reliable two-step MLE procedure and discuss differences to the CKM benchmark. Lastly, concerning the presentation and discussion of the results, we introduce a new measure that assesses the contribution rather than, as hitherto, the predictive power of each wedge to the considered fluctuations.

The data preparation of the application to the Great Recession in Germany shows a lack of homogeneity in the fluctuations of business investments and durables as well as net exports and government consumption. Thus, we separate those quantities. Likewise, the growth rates of most quantities are not homogeneous, which is why we detrend them separately. We augment the prototype economy by separating durable goods from business investments and net exports from government consumption. Additionally, we include long-run wedges to determine heterogeneous growth rates. Applying our MLE procedure, we detect two local maxima concerning the estimated structural parameters and identify the global one, for which we illustrate and discuss the results. We find that the efficiency

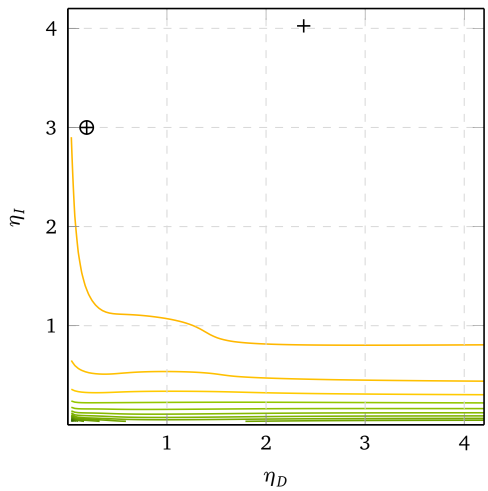
Figure 7: Adjustment costs specific wedge contribution



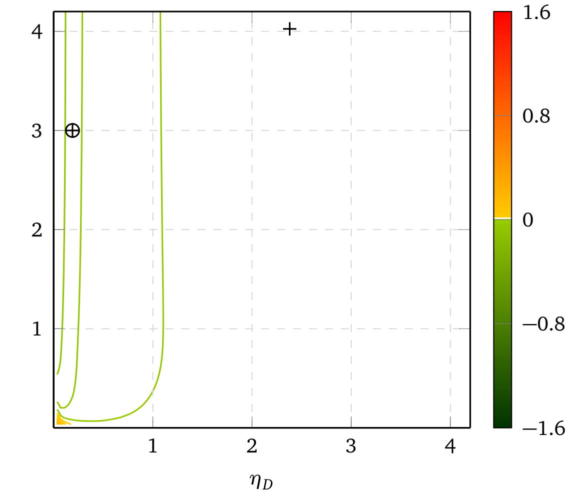
(a) Contribution of z_{At} to GDP ($\Delta_{z_{At}}$)



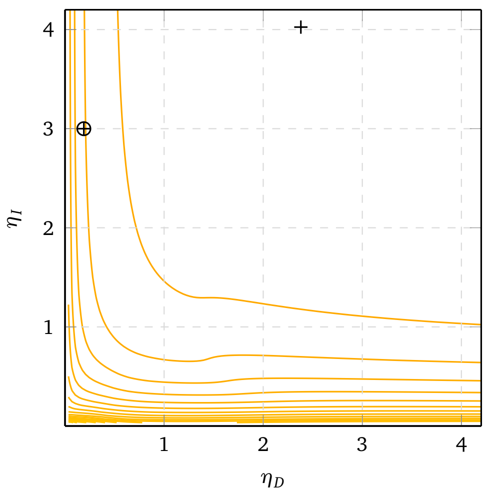
(b) Contribution of z_{Nt} to GDP ($\Delta_{z_{Nt}}$)



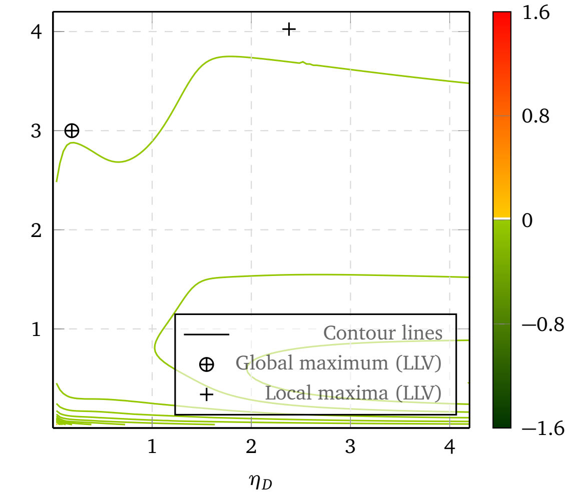
(c) Contribution of z_{It} to GDP ($\Delta_{z_{It}}$)



(d) Contribution of z_{Dt} to GDP ($\Delta_{z_{Dt}}$)

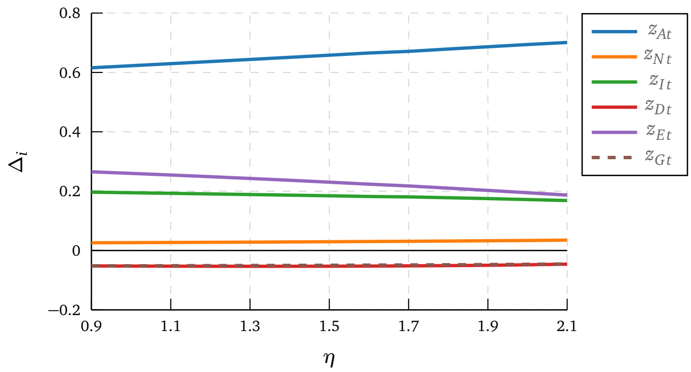


(e) Contribution of z_{Et} to GDP ($\Delta_{z_{Et}}$)



(f) Contribution of z_{Gt} to GDP ($\Delta_{z_{Gt}}$)

Figure 8: Inverse elasticity of intertemporal substitution specific wedge contribution



wedge, net exports, and investment wedge drove the recession. The durables and the government spending wedge acted counter-cyclical. We argue that the latter two collect parts of the fiscal stimulus program. The labor market wedge induced the recovery, which is mappable to expanded short-time work possibilities like the so-called Hartz reforms. We discuss our results against the literature and check their robustness to different choices of parameters that determine the elasticity of intertemporal substitution as well as capital and durables adjustment costs. Concerning the literature on BCA and the Great Recession in Germany, our results are not inconsistent with previous results, yet, insights are more detailed, and thereby indicate that previous studies underrate the negative impact of the investment wedge and, consequently, the role of the investment wedge equivalent financial frictions or distortions. Further, we quantify the effects of durable net-tax changes through the consumption Euler-equation inside a general equilibrium. We find substantial intertemporal substitution effects, making durables tax-changes an interesting tool of unconventional fiscal policy, which is coming more and more to a consensus. Our results are robust for all wedges except the investment wedge. However, the investments wedge's contribution to the Great Recession is positive in a wide range of the parameters' domains and is merely negative where the likelihood is the lowest.

The manual and its application should make BCA feasible in a standardized way. We hope that our estimation procedure will help to overcome problems regarding MLE. The original idea behind BCA is to provide researchers with an indicator that tells them where to introduce frictions into their models to replicate empirical business cycle fluctuations. Since the implementation is not an easy exercise, it mostly turned into whole projects. Hopefully, we have simplified BCA to the extent that it becomes a standard tool in the sections on stylized facts of a specific period of economic fluctuations or business cycles generally.

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APPENDIX

Refine and standardize Business Cycle Accounting using the example of the Great Recession in Germany

A Disaggregated Prototype Economy

A.1 Model

The following equations determine the model with stationary variables

$$y_t = k_{It}^{\alpha} (A_t N_t)^{1-\alpha}, \tag{A.1a}$$

$$r_t = \alpha \frac{y_t}{k_{It}}, \tag{A.1b}$$

$$w_t = (1-\alpha)\frac{y_t}{N_t}, \tag{A.1c}$$

$$\lambda_t = \phi c_t^{\phi(1-\eta)-1} k_{Dt}^{(1-\phi)(1-\eta)} (1-N_t)^{\psi(1-\eta)}, \tag{A.1d}$$

$$(1-\tau_{Nt}) = \frac{\psi}{\phi}\frac{c_t}{(1-N_t)w_t}, \tag{A.1e}$$

$$y_t = c_t + i_t + d_t + g_t + e_t, \tag{A.1f}$$

$$\mu_{It} = \lambda_t \frac{1+\tau_{It}}{1-\Theta'_{It}}, \tag{A.1g}$$

$$\mu_{Dt} = \lambda_t \frac{1+\tau_{Dt}}{1-\Theta'_{Dt}}, \tag{A.1h}$$

$$g_I \cdot \gamma_n k_{It+1} = (1-\delta_I)k_{It} + i_t - \Theta_{It} \cdot k_{It}, \tag{A.1i}$$

$$g_D \cdot \gamma_n k_{Dt+1} = (1-\delta_D)k_{Dt} + d_t - \Theta_{Dt} \cdot k_{Dt}, \tag{A.1j}$$

$$\mu_{It} = \beta g_{M_I} \mathbb{E}_t \left[\mu_{It+1}\left(1-\delta_I - \Theta_{It+1} + \frac{i_{t+1}}{k_{It+1}}\Theta'_{It+1}\right) + \lambda_{t+1} r_{t+1} \right], \tag{A.1k}$$

$$\mu_{Dt} = \beta g_{M_D} \mathbb{E}_t \left[\mu_{Dt+1}\left(1-\delta_D - \Theta_{Dt+1} + \frac{d_{t+1}}{k_{Dt+1}}\Theta'_{Dt+1}\right) + \lambda_{t+1} \frac{1-\phi}{\phi}\frac{c_{t+1}}{k_{Dt+1}} \right], \tag{A.1l}$$

with

$$g_{M_I} = g_Y^{\phi(1-\eta)} \cdot g_D^{(1-\phi)(1-\eta)} \cdot g_I^{-1},$$
$$g_{M_D} = g_Y^{\phi(1-\eta)} \cdot g_D^{(1-\phi)(1-\eta)-1},$$
$$\Theta_{Xt} = \frac{a_X}{2}\left(\frac{x_t}{k_{Xt}} - b_X\right)^2,$$
$$\Theta'_{Xt} = a_X\left(\frac{x_t}{k_{Xt}} - b_X\right),$$
$$b_X = x^*/k_X^*,$$

with $X \in \{I, D\}$, $x \in \{i, d\}$ and where * indicates the steady-state value. The fluctuation in the model is driven by the VAR(1)-process

$$\underbrace{\begin{bmatrix} \ln(z_{At+1}) \\ z_{Nt+1} \\ z_{It+1} \\ z_{Dt+1} \\ z_{Et+1} \\ \ln(z_{Gt+1}) \end{bmatrix}}_{:=\mathbf{z_{t+1}}} = \mathbf{\Pi} \underbrace{\begin{bmatrix} \ln(z_{At}) \\ z_{Nt} \\ z_{It} \\ z_{Dt} \\ z_{Et} \\ \ln(z_{Gt}) \end{bmatrix}}_{:=\mathbf{z_t}} + \underbrace{\begin{bmatrix} \epsilon_{At+1} \\ \epsilon_{Nt+1} \\ \epsilon_{It+1} \\ \epsilon_{Dt+1} \\ \epsilon_{Et+1} \\ \epsilon_{Gt+1} \end{bmatrix}}_{:=\boldsymbol{\epsilon}_{t+1}}, \qquad \boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}). \tag{A.2}$$

The stochastic process affects the wedges as follows

$$\begin{aligned} A_t &= A^* \cdot z_{At}, \\ \tau_{Nt} &= \tau_N^* + z_{Nt}, \\ \tau_{It} &= \tau_I^* + z_{It}, \\ \tau_{Dt} &= \tau_D^* + z_{Dt}, \\ e_t &= e^* + z_{Et}, \\ g_t &= g^* \cdot z_{Gt}. \end{aligned}$$

A.2 VAR Estimation

A.2.1 Mapping

Chari et al. (2007), Brinca et al. (2016), and various other authors map structural models into their prototype economy. Nutahara and Inaba (2012) apply BCA for misspecified wedges and find they are able to approximate the true wedges and the corresponding response of the agents adequately. We show first how to map the stimulus program to the prototype economy. Since the wedges' drivers are modeled as taxes, this is straightforward for most of the measures. Secondly, we reflect monetary policy.

Mapping the stimulus program

Government Wedge: We assign total government spending to the government spending wedge. These are mainly investments in infrastructure and financial support for local and state authority spending. Hence, the stimulus program increases the government wedge

Table A.1: Estimation of exogenous shock process

	Autoregressive Matrix						
Π	$\ln(s_A)$	s_N	s_I	s_D	s_E	$\ln(s_G)$	
$\ln(s_A)$	0.90	0.41	0.00	0.07	-0.21	-0.16	
s_N	0.01	0.83	0.01	-0.02	-0.12	-.01	
s_I	0.70	-1.71	0.96	-0.52	1.44	1.07	
s_D	0.27	-0.05	-0.00	0.66	0.16	-0.01	
s_E	0.06	-0.03	0.01	-0.05	0.62	-0.12	
$\ln(s_G)$	-0.05	0.17	-0.01	-0.05	-0.22	0.80	
	Correlation and standard errors						
$Corr(\epsilon_i,\epsilon_j)$	ϵ_A	ϵ_N	ϵ_I	ϵ_D	ϵ_E	ϵ_G	$100\cdot StD(\epsilon_i)$
ϵ_A	1.00						0.94
ϵ_N	0.03	1.00					0.34
ϵ_I	-0.49	-0.06	1.00				7.12
ϵ_D	0.27	-0.83	0.13	1.00			1.44
ϵ_E	0.31	0.70	-0.02	-0.36	1.00		0.59
ϵ_G	-0.10	0.13	-0.19	-0.16	-0.13	1.00	0.80

directly.

Durables Wedge: The two measures concerning new cars affect the durables wedge. For a given producer price, both measures reduce the absolute tax or the relative price of durables from the household's perspective. Hence, they increase the durables wedge.

Investment Wedge: The first part of the stimulus program which affects the investment wedge is a subsidy for investments in innovations. The second is an increased tax deduction by allowing for a reducing-balance method. For given producer prices, absolute taxes or the relative price of investment decreases, and thus the investment wedge increases.

Chari et al. (2007) show how to map financial frictions in terms of a financial accelerator and Brinca et al. (2016) show how to map financial frictions in terms of collateral constraints into a prototype economy with an investment wedge. The loan and guarantee program lowers financial frictions, in particular, they mitigate the banks' collateral constraints. Following this, the loan and guarantee program also raises the investment wedge.

Labor Wedge: The stimulus program lowers income tax and social contribution, this increases the labor wedge in general.

Brinca et al. (2016) show the link between a prototype economy with efficiency and labor wedges and an economy with search and matching frictions. The mentioned labor market actions, e.g., expanded short-time work, reduce such frictions and thus, increase the labor market wedge. The effects should be delayed in time due to lower hiring frictions in the aftermath of the crisis.

Efficiency Wedge: Due to the labor market actions in the previous paragraph, the efficiency wedge increases also due to better matching. Further, the expanded short-time work possibilities reduce labor hoarding, since the firm can both retain employees to lower future hiring frictions and adjust hours worked. As a consequence, the efficiency wedge increases.

As shown by Chari et al. (2007), input-financing frictions are associated with efficiency wedges. These frictions appear when firms must borrow for an input good and some firms are financially more constrained than others. Such firms have to pay higher interest rates. The loan and guarantee program lowers financial constraints and thus increases the efficiency wedge.

Net exports: The increase in Hermes coverage advances the conditions for exports. Nevertheless, the effects are probably only rather small.

Mapping monetary policy

Government Wedge: Quantitative easing lowers the bonds' interest rates and this lowers the costs of debt-financed government spending, which may indirectly increase the government wedge.

Durables Wedge: Since refinancing is cheaper, for a given real rate of return, investment increases. Hence, monetary policy changes the intertemporal decision of a household, which is reflected in a higher durables wedge. Furthermore, the provision of liquidity also changes the intertemporal decisions of liquidity-constrained households, which also reflects in a higher durables wedge.

Investment Wedge: Both mentioned effects of the durables wedge have the same effect on the investment wedge. The provision of liquidity and cheaper refinancing lower frictions in the investment market.

As already mentioned, Brinca et al. (2016) show how to map an economy with collateral-constrained banks into a prototype economy with an investment wedge. Lower collateral constraints lower frictions in the investment market. Thus, the slacked collateral requirements by the ECB increase the investment wedge.

Efficiency Wedge: As mentioned above, input-financing frictions are associated with efficiency wedges (see Chari et al., 2007). The friction appears when firms must borrow for input goods and some firms are financially more constrained than others. Those firms have to pay higher interest rates. The Security Markets Program can lower these frictions and thus, increases efficiency.

A.3 Determining $\mathbf{Z}_1$ and $\mathbf{Z}_2$

Here we calculate $\mathbf{Z}_1$ and $\mathbf{Z}_2$ from eq. (3a) to show that the eigenvalues of $\mathbf{Z}_2$ are inside the unit circle. We start by deriving the log-linearized version of equations (A.1i) and (A.1j):

$$\Rightarrow \hat{k}_{Xt+1} = \frac{1-\Theta_X'^*}{(1-\delta_X)k_X^* + x^* - k_X^*\Theta_X^*}(x_t - x^*) + \frac{1-\delta_X - \Theta_X^* + \Theta_X'^*\frac{x^*}{k^*}}{(1-\delta_X)k_X^* + x^* - k_X^*\Theta_X^*}(k_t - k^*),$$

with $\Theta_X^* = \Theta_X'^* = 0$

and $g_X\gamma_n k_X^* = (1-\delta_X)k_X^* + x^* \Leftrightarrow (g_X\gamma_n - 1 + \delta_X) = \frac{x^*}{k_X^*}$,

$$\Rightarrow \hat{k}_{Xt+1} = \frac{g_X\gamma_n - 1 + \delta_X}{g_X\gamma_n}\hat{x}_t + \frac{1-\delta_X}{g_X\gamma_n}\hat{k}_{xt}$$

The log-linearized version implies that

$$\mathbf{x_t} = \begin{bmatrix} 0 & 0 & \frac{g_I\gamma_n - 1 + \delta_I}{g_I\gamma_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g_D\gamma_n - 1 + \delta_D}{g_D\gamma_n} & 0 & 0 \end{bmatrix}\mathbf{y_{t-1}} + \begin{bmatrix} \frac{1-\delta_I}{g_I\gamma_n} & 0 \\ 0 & \frac{1-\delta_D}{g_D\gamma_n} \end{bmatrix}\mathbf{x_{t-1}}$$

is a solution of equation (3a). Further, as the solution $\mathbf{L}_\cdot^\cdot$ is unique, $\mathbf{Z}_1 = \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1}$ and $\mathbf{Z}_2 = \mathbf{L}_\mathbf{x}^\mathbf{x} - \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1}\mathbf{L}_\mathbf{x}^\mathbf{y}$ are unique and thus,

$$\mathbf{Z}_1 = \begin{bmatrix} 0 & 0 & \frac{g_I\gamma_n - 1 + \delta_I}{g_I\gamma_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g_D\gamma_n - 1 + \delta_D}{g_D\gamma_n} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Z}_2 = \begin{bmatrix} \frac{1-\delta_I}{g_I\gamma_n} & 0 \\ 0 & \frac{1-\delta_D}{g_D\gamma_n} \end{bmatrix}$$

are the unique solution for $\mathbf{Z}_1$ and $\mathbf{Z}_2$. Furthermore, as long as $(1-\delta_x) < \gamma_x\gamma_n$ holds, the eigenvalues of $\mathbf{Z}_2$ are inside the unit circle.

B Chari et al. (2007) Prototype Economy

B.1 Model

The per period utility of the representative household is parameterized as follows

$$u(C_t, D_t, N_t) = \begin{cases} \ln(C_t) + \psi \ln(1 - N_t) & \text{for } \eta = 1, \\ \frac{\left(C_t \cdot (1-N_t)^{\psi}\right)^{1-\eta} - 1}{1-\eta} & \text{for } \eta \neq 1, \end{cases} \tag{B.3}$$

where C_t denotes consumption of non-durable goods and N_t is the household's labor supply.

The household maximizes its expected life-time-utility

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \gamma_n)^s u(C_{t+s}, N_{t+s}) \tag{B.4}$$

with respect to consumption and labor, with γ_n as the population growth factor, and subject to the budget constraint

$$C_t + (1 + \tau_{It}) P_{It} I_t \leq R_t K_t + (1 - \tau_{Nt}) W_t N_t + T_t, \tag{B.5}$$

where K_t denotes the productive capital stock (capital stock hereafter), I_t investment in capital, T_t lump-sum transfers, R_t the rental rate on capital, and W_t the real wage. The tax rates τ_{Nt} and τ_{It} are used to model wedges in the labor and investment market. P_{It} is the relative prices for investment and reflects the wedges' long-run element. The consumption good is the numeraire. The capital stock follows the law-of-motion

$$\gamma_n K_{It+1} = (1 - \delta_I) K_t + I_t - \Theta_{It}\left(\frac{I_t}{K_t}\right) K_t, \; \Theta_{It}\left(\frac{I_t}{K_t}\right) = \frac{a_I}{2}\left(\frac{I_t}{K_t} - b_I\right)^2, \tag{B.6}$$

with b_I as the investment-to-capital ratio in the long run.

The representative firm faces perfect competition, produces output Y_t, which can be used for (government) consumption and investment, and maximizes its profits

$$Y_t - W_t N_t - R_t K_t \tag{B.7}$$

with respect to capital and labor and subject to the output's Y_t production function

$$Y_t = K_t^{\alpha}(\gamma_a^t A_t N_t)^{1-\alpha}. \tag{B.8}$$

The parameter γ_a denotes the growth factor of labor augmenting technical progress and A_t the efficiency wedge.

The government expenditures in consumption units $P_{Gt}G_t$ are exogenous and the government chooses lump-sum transfers T_t, so that its budget constraint

$$P_{Gt}G_t + T_t \leq \tau_{Nt}W_tN_t + \tau_{It}P_{It}I_t \tag{B.9}$$

always binds. Thereby, the resource constraint of the economy is

$$Y_t = C_t + P_{It}I_t + P_{Gt}G_t. \tag{B.10}$$

Growth component: As already mentioned, the population grows with γ_n and technical progress with γ_a. Furthermore, the wedges evolve differently, which the relative prices reflect. In the long run $P_{Xt} \in \{P_{It}, P_{Gt}\}$ evolves with $P_{Xt} = g_{P_X} P_{Xt-1}$. The ensuing trend growth factors of different variables X_t are described in Table 4. These variables are scaled by $x_t = \frac{X_t}{g_X^t}$ and are thus stationary variables.

Table B.2: Growth factors

X_t	Y_t	C_t	W_t	T_t	I_t	K_t	R_t	G_t	γ_a	N_t	P_{Xt}
g_X	g_Y	g_Y	g_Y	g_Y	g_I	g_I	g_Y/g_I	g_G	$g_Y^{\frac{1}{1-\alpha}} g_I^{\frac{\alpha}{\alpha-1}}$	1	$g_{P_X} = \frac{g_Y}{g_X}$

Business cycle component: The VAR(1)-process

$$\mathbf{z}_{t+1} = \mathbf{\Pi}\,\mathbf{z}_t + \boldsymbol{\epsilon}_{t+1},\ \boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \mathbf{\Sigma}), \tag{B.11}$$

drives the fluctuation of the model, where $\mathbf{z}_t = \begin{pmatrix}\ln(z_{At}) & z_{Nt} & z_{It} & \ln(z_{Gt})\end{pmatrix}^T$ and $\boldsymbol{\epsilon}_t = \begin{pmatrix}\epsilon_{At} & \epsilon_{Nt} & \epsilon_{It} & \epsilon_{Gt}\end{pmatrix}^T$ and $\mathbf{\Sigma}$ is positive definite. The stochastic process affects the wedges as follows

$$A_t = A^* \cdot z_{At}, \qquad \tau_{Nt} = \tau_N^* + z_{Nt}, \qquad \tau_{It} = \tau_I^* + z_{It}, \qquad g_t = g^* \cdot z_{Gt},$$

where A^*, τ_N^*, τ_I^*, and g^* are the corresponding steady-state component of the different distortions. Chari et al. (2007) define the four time-varying wedges as follows: The efficiency wedge A_t, the government spending wedge g_t, the labor wedge $1-\tau_{Nt}$, and the investment wedge $\frac{1}{1+\tau_{It}}$. The latter is defined so that, similar to the labor market wedge, increases act like subsidies, and decreases act like taxes in comparison to the steady-state value. Since the cyclical component includes the steady-state component, detrended prices p_{Gt}, p_{It} are normed to one. We present in the following the full dynamic equilibrium of the model in stationary variables.

$$y_t = k_t^{\alpha}(A_t N_t)^{1-\alpha}, \tag{B.12a}$$

$$r_t = \alpha \frac{y_t}{k_t}, \tag{B.12b}$$

$$w_t = (1-\alpha)\frac{y_t}{N_t}, \tag{B.12c}$$

$$\lambda_t = c_t^{(1-\eta)-1}(1-N_t)^{\psi(1-\eta)}, \tag{B.12d}$$

$$(1-\tau_{Nt}) = \psi \frac{c_t}{(1-N_t)w_t}, \tag{B.12e}$$

$$y_t = c_t + i_t + g_t, \tag{B.12f}$$

$$\mu_{It} = \lambda_t \frac{1+\tau_{It}}{1-\Theta'_{It}}, \tag{B.12g}$$

$$g_I \cdot \gamma_n k_{t+1} = (1-\delta_I)k_t + i_t - \Theta_{It} \cdot k_t, \tag{B.12h}$$

$$\mu_{It} = \beta g_{M_I} \mathbb{E}_t \left[\mu_{It+1} \left(1 - \delta_I - \Theta_{It+1} + \frac{i_{t+1}}{k_{t+1}} \Theta'_{It+1} \right) + \lambda_{t+1} r_{t+1} \right], \tag{B.12i}$$

with

$$g_{M_I} = g_Y^{1-\eta} \cdot g_I^{-1},$$

$$\Theta_{It} = \frac{a_I}{2}\left(\frac{i_t}{k_t} - b_I\right)^2,$$

$$\Theta'_{It} = a_I\left(\frac{i_t}{k_t} - b_I\right),$$

$$b_I = i^*/k^*,$$

where * indicates the steady-state value. Further,

$$\underbrace{\begin{bmatrix} \ln(z_{At+1}) \\ z_{Nt+1} \\ z_{It+1} \\ \ln(z_{Gt+1}) \end{bmatrix}}_{:=\mathbf{z_{t+1}}} = \mathbf{\Pi} \underbrace{\begin{bmatrix} \ln(z_{At}) \\ z_{Nt} \\ z_{It} \\ \ln(z_{Gt}) \end{bmatrix}}_{:=\mathbf{z_t}} + \underbrace{\begin{bmatrix} \epsilon_{At+1} \\ \epsilon_{Nt+1} \\ \epsilon_{It+1} \\ \epsilon_{Gt+1} \end{bmatrix}}_{:=\boldsymbol{\epsilon}_{t+1}}, \qquad \boldsymbol{\epsilon_t} \sim N(\mathbf{0}, \mathbf{\Sigma}), \tag{B.13}$$

$$A_t = A^* \cdot z_{At}, \tag{B.14}$$

$$\tau_{Nt} = \tau_N^* + z_{Nt}, \tag{B.15}$$

$$\tau_{It} = \tau_I^* + z_{It}, \tag{B.16}$$

$$g_t = g^* \cdot z_{Gt}. \tag{B.17}$$

B.2 Observables and data manipulation

The vector of observables reads as follows $\mathbf{y}_t = \begin{pmatrix} \hat{y}_t & \hat{N}_t & \hat{i}_t & \hat{g}_t \end{pmatrix}^T$. In contrast to our modified model government consumption is the sum of government consumption and net exports and investments are the sum of durables and investments.

B.3 Calibration and estimation

The calibration and estimation strategy is similar to our modified model. We estimate the elasticity of the price of capital η_I as well as the parameters of the stochastic process. All other parameters are calibrated and the long-run ratios are pinned down to their long-run averages. Tables B.3 and B.4 present all relevant parameters.

Table B.3: Calibration and growth accounting for the Chari et al. (2007) economy

Parameter	Description	Value
α	Capital share	0.34
β	Discount factor	0.994
δ_I	Rate of capital depreciation	0.0203
ψ	Preference weight of labor	2.24
η	Risk aversion	1
η_I	Elasticity of the price of capital	0.86
$\ln(\gamma_n^4)$	Annual growth rate of population	0.03%
$\ln(g_Y^4)$	Annual growth rate of GDP	1.32%
$\ln(g_I^4)$	Annual growth rate of investment	0.79%

Table B.4: Estimation of exogenous shock process of the Chari et al. (2007) economy

	Autoregressive Matrix				
Π	$ln(s_A)$	s_N	s_I	$ln(s_G)$	
$ln(s_A)$	0.93	0.09	0.05	−0.03	
s_N	−0.01	0.73	0.04	−0.00	
s_I	0.03	2.03	0.67	−0.02	
$ln(s_G)$	0.09	−1.17	0.08	0.84	
	Correlation and standard errors				
$Corr(\epsilon_i, \epsilon_j)$	ϵ_A	ϵ_N	ϵ_I	ϵ_G	$100 \cdot StD(\epsilon_i)$
ϵ_A	1.00				0.94
ϵ_N	0.21	1.00			0.29
ϵ_I	−0.27	−0.61	1.00		1.77
ϵ_G	0.43	0.77	−0.34	1.00	2.71

C Data

GDP, investment, durables, government expenditures, hours worked and net exports to GDP are the observables. A regression with the logarithm of the first four observables as dependent variables and time as independent variable provides the necessary components. The coefficient approximates the growth rate for growth accounting and the residuals the relative deviation from balanced growth. The latter is used for business cycle accounting, the former for growth accounting. Hours worked are the relative deviation from average. Negative values for net exports prevent logarithmization. A regression with net exports relative to GDP as dependent variable and time as independent variable provides an auxiliary variable. The coefficient is the excess growth rate of net exports compared to GDP growth. The residuals are the deviation from the long-run net exports to GDP rate, which is computable in the model.

The data are taken from the Fachserie 18: National accounts, domestic product from the German Federal Statistical Office.

- **Pop:** Total Population 1991:I-2018:I

  Source: 2.1.7 Population and labour force participation 1; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

- **Hours:** Hours worked by persons in employment 1991:I-2018:I

  Source: 2.1.8 Persons in employment, employees and hours worked (domestic concept) 2; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

- **GDP:** 1991:I-2018:I

  Nominal source: 2.3.1 Use of gross domestic product at current prices 2; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

  Real source: 2.3.2 Use of gross domestic product, price-adjusted 2; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

- **PCE:** Private Consumption Expenditures of households 1991:I-2018:I

  Nominal source: 2.3.3 Final consumption expenditure at current prices 3; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

  Real source: 2.3.4 Final consumption expenditure at , price-adjusted; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

- **Govern. Consumption:** Government final consumption expenditure (domestic use) 1991:I-2018:I

Nominal source: 2.3.3 Final consumption expenditure at current prices 3; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

Real source: 2.3.4 Final consumption expenditure at , price-adjusted; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

- **Investment:** Gross fixed capital formation 1991:I-2018:I

  Nominal source: 2.3.1 gross fixed capital formation at current prices 2; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

  Real source: 2.3.2 gross fixed capital formation, price-adjusted 2; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

- **Net Exports:** Balance of exports and imports 1991:I-2018:I

  Nominal source: 2.3.1 Balance of exports and imports at current prices 2; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

  Real source: 2.3.2 Balance of exports and imports, price-adjusted 2; Seasonally adjusted quarterly results using Census X-12-ARIMA and BV4.1 - Fachserie 18 Reihe 1.3 - 1st Quarter 2018

- **Durables:** Langlebige Güter (Durable Goods) 1991:I-2018:I

  Nominal source: 2.14 Konsumausgaben der privaten Haushalte im Inland nach Dauerhaftigkeit der Güter, Saison- und kalenderbereinigt in jeweiligen Preisen 4; Private Konsumausgaben und Verfügbares Einkommen - 1. Vierteljahr 2018

  Real source: 2.14 Konsumausgaben der privaten Haushalte im Inland nach Dauerhaftigkeit der Güter, Saison- und kalenderbereinigt - preisbereinigt 4; Private Konsumausgaben und Verfügbares Einkommen - 1. Vierteljahr 2018

  (available in German only: Domestic consumer spending on durable goods, seasonally and calendar adjusted 4; Private consumption expenditure and disposable income - 1st quarter of 2018)

D Solving for the Policy function of the linearized Model

In this section of the appendix, we illustrate how to solve for the policy function of the (log-)linearized model. Thereby we follow closely to the proceedings suggested by Uhlig (1999) and Christiano (2002).

Assume we have a linear rational expectations model of the form

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_{x,t}^{(d)} & \mathbf{g}_{u,t}^{(d)} & \mathbf{g}_{z,t}^{(d)} & \mathbf{g}_{x,t+1}^{(d)} & \mathbf{0} & \mathbf{0} \\ \mathbf{g}_{x,t}^{(s)} & \mathbf{g}_{u,t}^{(s)} & \mathbf{g}_{z,t}^{(s)} & \mathbf{g}_{x,t+1}^{(s)} & \mathbf{g}_{u,t+1}^{(s)} & \mathbf{g}_{z,t+1}^{(s)} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Pi} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{x}_t \\ \mathbf{u}_t \\ \mathbf{z}_t \\ \mathbf{x}_{t+1} \\ \mathbb{E}_t\left[\mathbf{u}_{t+1}\right] \\ \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] \end{pmatrix}, \quad \text{with} \quad \begin{array}{lcl} \mathbf{g}_{x,t}^{(d)}, \mathbf{g}_{x,t+1}^{(d)} & \in & \mathbb{R}^{n_u \times n_x}, \\ \mathbf{g}_{u,t}^{(d)} & \in & \mathbb{R}^{n_u \times n_u}, \\ \mathbf{g}_{z,t}^{(d)}, & \in & \mathbb{R}^{n_u \times n_z}, \\ \mathbf{g}_{x,t}^{(s)}, \mathbf{g}_{x,t+1}^{(s)} & \in & \mathbb{R}^{n_x \times n_x}, \\ \mathbf{g}_{u,t}^{(s)}, \mathbf{g}_{u,t+1}^{(s)} & \in & \mathbb{R}^{n_x \times n_u}, \\ \mathbf{g}_{z,t}^{(s)}, \mathbf{g}_{z,t+1}^{(s)} & \in & \mathbb{R}^{n_x \times n_z}, \\ \mathbf{\Pi} & \in & \mathbb{R}^{n_z \times n_z}, \end{array} , \tag{D.18}$$

where $\mathbf{x}_t \in \mathbb{R}^{n_x}$ denote the predetermined state variables, $\mathbf{u}_t \in \mathbb{R}^{n_u}$ denote the set of endogenous (jump) variables and $\mathbf{z}_t \in \mathbb{R}^{n_z}$ are the exogenous state variables of the model. If the square matrix $\mathbf{g}_{u,t}^{(d)}$ has full rank, we can write $\mathbf{u}_t$ as a linear function

$$\mathbf{u}_t = \mathbf{h}_{x,t}\ \mathbf{x}_t + \mathbf{h}_{x,t+1}\ \mathbf{x}_{t+1} + \mathbf{h}_{z,t}\ \mathbf{z}_t \tag{D.19}$$

of $\mathbf{x}_t$, $\mathbf{x}_{t+1}$ and $\mathbf{z}_t$ with

$$\mathbf{h}_{x,t} := -\left(\mathbf{g}_{u,t}^{(d)}\right)^{-1} \mathbf{g}_{x,t}^{(d)}, \qquad \mathbf{h}_{x,t+1} := -\left(\mathbf{g}_{u,t}^{(d)}\right)^{-1} \mathbf{g}_{x,t+1}^{(d)}, \qquad \mathbf{h}_{z,t} := -\left(\mathbf{g}_{u,t}^{(d)}\right)^{-1} \mathbf{g}_{z,t}^{(d)}.$$

Substituting (D.19) in the second row of (D.18) yields:

$$\begin{aligned} \mathbf{0} &= \mathbf{g}_{x,t}^{(s)}\ \mathbf{x}_t + \mathbf{g}_{u,t}^{(s)}\ \mathbf{h}_{x,t}\ \mathbf{x}_t + \mathbf{g}_{u,t}^{(s)}\ \mathbf{h}_{x,t+1}\ \mathbf{x}_{t+1} + \mathbf{g}_{u,t}^{(s)}\ \mathbf{h}_{z,t}\ \mathbf{z}_t + \mathbf{g}_{z,t}^{(s)}\ \mathbf{z}_t + \mathbf{g}_{x,t+1}^{(s)}\ \mathbf{x}_{t+1} \\ &\quad + \mathbf{g}_{u,t+1}^{(s)}\ \mathbf{h}_{x,t}\ \mathbf{x}_{t+1} + \mathbf{g}_{u,t+1}^{(s)}\ \mathbf{h}_{x,t+1}\ \mathbb{E}_t\left[\mathbf{x}_{t+2}\right] + \mathbf{g}_{u,t+1}^{(s)}\ \mathbf{h}_{z,t}\ \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] + \mathbf{g}_{z,t+1}^{(s)}\ \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] \\ &= \left(\mathbf{g}_{x,t}^{(s)} + \mathbf{g}_{u,t}^{(s)}\ \mathbf{h}_{x,t}\right) \mathbf{x}_t + \left(\mathbf{g}_{x,t+1}^{(s)} + \mathbf{g}_{u,t}^{(s)}\ \mathbf{h}_{x,t+1} + \mathbf{g}_{u,t+1}^{(s)}\ \mathbf{h}_{x,t}\right) \mathbf{x}_{t+1} + \left(\mathbf{g}_{u,t+1}^{(s)}\ \mathbf{h}_{x,t+1}\right) \mathbb{E}_t\left[\mathbf{x}_{t+2}\right] \\ &\quad + \left(\mathbf{g}_{z,t}^{(s)} + \mathbf{g}_{u,t}^{(s)}\ \mathbf{h}_{z,t}\right) \mathbf{z}_t + \left(\mathbf{g}_{z,t+1}^{(s)} + \mathbf{g}_{u,t+1}^{(s)}\ \mathbf{h}_{z,t}\right) \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] \\ &= \mathbf{A}_2\ \mathbf{x}_t + \mathbf{A}_1\ \mathbf{x}_{t+1} + \mathbf{A}_0\ \mathbb{E}_t\left[\mathbf{x}_{t+2}\right] + \mathbf{B}_1\ \mathbf{z}_t + \mathbf{B}_0\ \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] \end{aligned} \tag{D.20}$$

with

$$\mathbf{A}_0 := \mathbf{g}^{(s)}_{u,t+1}\ \mathbf{h}_{x,t+1}, \qquad \mathbf{A}_1 := \mathbf{g}^{(s)}_{x,t+1} + \mathbf{g}^{(s)}_{u,t}\ \mathbf{h}_{x,t+1} + \mathbf{g}^{(s)}_{u,t+1}\ \mathbf{h}_{x,t}, \quad \mathbf{A}_2 := \mathbf{g}^{(s)}_{x,t} + \mathbf{g}^{(s)}_{u,t}\ \mathbf{h}_{x,t},$$
$$\mathbf{B}_0 := \mathbf{g}^{(s)}_{z,t+1} + \mathbf{g}^{(s)}_{u,t+1}\ \mathbf{h}_{z,t}, \quad \mathbf{B}_1 := \mathbf{g}^{(s)}_{z,t} + \mathbf{g}^{(s)}_{u,t}\ \mathbf{h}_{z,t}.$$

If the solution of (D.20) is given by

$$\mathbf{x}_{t+1} = \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{x}_t + \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_t \tag{D.21}$$

we may rewrite (D.20) to

$$\begin{aligned}
\mathbf{0} &= \mathbf{A}_2\ \mathbf{x}_t + \mathbf{A}_1\ \left(\mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{x}_t + \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_t\right) + \mathbf{A}_0\ \mathbb{E}_t\left[\left(\mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \left(\mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{x}_t + \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_t\right) + \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_{t+1}\right)\right] + \mathbf{B}_1\ \mathbf{z}_t + \mathbf{B}_0\ \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] \\
&= \mathbf{A}_2\ \mathbf{x}_t + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{x}_t + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_t + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \left(\mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{x}_t + \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_t\right) + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] + \mathbf{B}_1\ \mathbf{z}_t + \mathbf{B}_0\ \mathbb{E}_t\left[\mathbf{z}_{t+1}\right] \\
&= \mathbf{A}_2\ \mathbf{x}_t + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{x}_t + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_t + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{x}_t + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{z}_t + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{\Pi}\ \mathbf{z}_t + \mathbf{B}_1\ \mathbf{z}_t + \mathbf{B}_0\ \mathbf{\Pi}\ \mathbf{z}_t \\
&= \left(\mathbf{A}_0\ (\mathbf{L}^{\mathbf{x}}_{\mathbf{x}})^2 + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}} + \mathbf{A}_2\right)\ \mathbf{x}_t + \left(\mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}} + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{\Pi} + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}} + \mathbf{B}_0\ \mathbf{\Pi} + \mathbf{B}_1\right)\ \mathbf{z}_t
\end{aligned} \tag{D.22}$$

Since (D.22) must hold for all $\mathbf{x}_t$ and $\mathbf{z}_t$, by setting $\mathbf{z}_t = \mathbf{0}$ we can obtain $\mathbf{L}^{\mathbf{x}}_{\mathbf{x}}$ as the stable solution of the quadratic matrix equation

$$\mathbf{0} = \mathbf{A}_0\ (\mathbf{L}^{\mathbf{x}}_{\mathbf{x}})^2 + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}} + \mathbf{A}_2. \tag{D.23}$$

The proceeding to solve (D.23) using generalized eigenvalues is provided by Uhlig (1999). We instead apply the method of Klein (2000) and use the QZ-Decomposition to solve system

$$\begin{pmatrix} -\mathbf{A}_1 & -\mathbf{A}_0 \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}}_{t+1} \\ \tilde{\mathbf{x}}_{t+2} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}}_{t} \\ \tilde{\mathbf{x}}_{t+1} \end{pmatrix} \tag{D.24}$$

for it's stable solution $\tilde{\mathbf{x}}_{t+1} = \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \tilde{\mathbf{x}}_t$. To obtain the $\mathbf{L}^{\mathbf{x}}_{\mathbf{z}}$, we again use the fact that (D.22) must hold for all $\mathbf{x}_t$ and $\mathbf{z}_t$. Setting $\mathbf{x}_t = \mathbf{0}$ we receive

$$\begin{aligned}
\mathbf{0} &= \left(\mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}}\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}} + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{\Pi} + \mathbf{A}_1\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}} + \mathbf{B}_0\ \mathbf{\Pi} + \mathbf{B}_1\right) \\
&= \left(\mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}} + \mathbf{A}_1\right)\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}} + \mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\ \mathbf{\Pi} + \mathbf{B}_0\ \mathbf{\Pi} + \mathbf{B}_1 && \text{(D.25)} \\
&\Rightarrow \operatorname{vec}\left(\mathbf{L}^{\mathbf{x}}_{\mathbf{z}}\right) = -\left[\mathbf{I} \otimes \left(\mathbf{A}_0\ \mathbf{L}^{\mathbf{x}}_{\mathbf{x}} + \mathbf{A}_1\right) + \mathbf{\Pi}^T \otimes \mathbf{A}_0\right]^{-1} \operatorname{vec}\left(\mathbf{B}_0\ \mathbf{\Pi} + \mathbf{B}_1\right) && \text{(D.26)}
\end{aligned}$$

In a last step we may use (D.19) to define the matrices

$$\mathbf{L}_{\mathbf{x}}^{\mathbf{u}} = \mathbf{h}_{x,t} + \mathbf{h}_{x,t+1}\mathbf{L}_{\mathbf{x}}^{\mathbf{x}} \tag{D.27}$$

$$\mathbf{L}_{\mathbf{z}}^{\mathbf{u}} = \mathbf{h}_{z,t} + \mathbf{h}_{x,t+1}\, \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} \tag{D.28}$$

so that

$$\begin{aligned} \mathbf{u}_t &= \mathbf{h}_{x,t}\, \mathbf{x}_t + \mathbf{h}_{x,t+1}\, \mathbf{x}_{t+1} + \mathbf{h}_{z,t}\, \mathbf{z}_t \\ &= \mathbf{h}_{x,t}\, \mathbf{x}_t + \mathbf{h}_{x,t+1}\, \mathbf{L}_{\mathbf{x}}^{\mathbf{x}}\, \mathbf{x}_t + \mathbf{h}_{x,t+1}\, \mathbf{L}_{\mathbf{z}}^{\mathbf{x}}\, \mathbf{z}_t + \mathbf{h}_{z,t}\, \mathbf{z}_t \\ &= \mathbf{L}_{\mathbf{x}}^{\mathbf{u}}\, \mathbf{x}_t + \mathbf{L}_{\mathbf{z}}^{\mathbf{u}}\, \mathbf{z}_t. \end{aligned} \tag{D.29}$$

Moreover, we can arrange the set of endogenous variables $\mathbf{u}_t$ such that the first n_z entries correspond to the observable variables in our estimation procedure. Thus, if we define $\mathbf{y}_t \in \mathbb{R}^{n_z}$, $\mathbf{c}_t \in \mathbb{R}^{n_u - n_z}$, $\mathbf{h}_{x,t}^{y} \in \mathbb{R}^{n_z \times n_x}$, $\mathbf{h}_{x,t}^{c} \in \mathbb{R}^{n_u - n_z \times n_x}$, $\mathbf{h}_{x,t+1}^{y} \in \mathbb{R}^{n_z \times n_x}$, $\mathbf{h}_{x,t+1}^{c} \in \mathbb{R}^{n_u - n_z \times n_x}$, $\mathbf{h}_{z,t}^{y} \in \mathbb{R}^{n_z \times n_x}$ and $\mathbf{h}_{z,t}^{c} \in \mathbb{R}^{n_u - n_z \times n_x}$ with

$$\mathbf{u}_t = \begin{pmatrix} \mathbf{y}_t \\ \mathbf{c}_t \end{pmatrix}, \qquad \mathbf{h}_{x,t} = \begin{pmatrix} \mathbf{h}_{x,t}^{y} \\ \mathbf{h}_{x,t}^{c} \end{pmatrix}, \qquad \mathbf{h}_{z,t} = \begin{pmatrix} \mathbf{h}_{z,t}^{y} \\ \mathbf{h}_{z,t}^{c} \end{pmatrix}, \qquad \mathbf{h}_{x,t+1} = \begin{pmatrix} \mathbf{h}_{x,t+1}^{y} \\ \mathbf{h}_{x,t+1}^{c} \end{pmatrix},$$

we may rewrite (D.27) and (D.28) to

$$\begin{aligned} \mathbf{u}_t &= \mathbf{h}_{x,t}\, \mathbf{x}_t + \mathbf{h}_{x,t+1}\, \mathbf{x}_{t+1} + \mathbf{h}_{z,t}\, \mathbf{z}_t \\ &= \mathbf{h}_{x,t}\, \mathbf{x}_t + \mathbf{h}_{x,t+1}\, \mathbf{L}_{\mathbf{x}}^{\mathbf{x}}\, \mathbf{x}_t + \mathbf{h}_{z,t}\, \mathbf{z}_t + \mathbf{h}_{x,t+1}\, \mathbf{L}_{\mathbf{z}}^{\mathbf{x}}\, \mathbf{z}_t \\ &= \left(\begin{pmatrix} \mathbf{h}_{x,t}^{y} \\ \mathbf{h}_{x,t}^{c} \end{pmatrix} + \begin{pmatrix} \mathbf{h}_{x,t+1}^{y} \\ \mathbf{h}_{x,t+1}^{c} \end{pmatrix} \mathbf{L}_{\mathbf{x}}^{\mathbf{x}} \right) \mathbf{x}_t + \left(\begin{pmatrix} \mathbf{h}_{z,t}^{y} \\ \mathbf{h}_{z,t}^{c} \end{pmatrix} + \begin{pmatrix} \mathbf{h}_{x,t+1}^{y} \\ \mathbf{h}_{x,t+1}^{c} \end{pmatrix} \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} \right) \mathbf{z}_t \\ &= \begin{pmatrix} \mathbf{h}_{x,t}^{y} + \mathbf{h}_{x,t+1}^{y} \mathbf{L}_{\mathbf{x}}^{\mathbf{x}} \\ \mathbf{h}_{x,t}^{c} + \mathbf{h}_{x,t+1}^{c}\, \mathbf{L}_{\mathbf{x}}^{\mathbf{x}} \end{pmatrix} \mathbf{x}_t + \begin{pmatrix} \mathbf{h}_{z,t}^{y} + \mathbf{h}_{x,t+1}^{y}\, \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} \\ \mathbf{h}_{z,t}^{c} + \mathbf{h}_{x,t+1}^{c}\, \mathbf{L}_{\mathbf{z}}^{\mathbf{x}} \end{pmatrix} \mathbf{z}_t \\ &= \begin{pmatrix} \mathbf{L}_{\mathbf{x}}^{\mathbf{y}} \\ \mathbf{L}_{\mathbf{x}}^{\mathbf{c}} \end{pmatrix} \mathbf{x}_t + \begin{pmatrix} \mathbf{L}_{\mathbf{z}}^{\mathbf{y}} \\ \mathbf{L}_{\mathbf{z}}^{\mathbf{c}} \end{pmatrix} \mathbf{z}_t. \end{aligned}$$

with

$$\mathbf{L}_{\mathbf{x}}^{\mathbf{y}} = \mathbf{h}_{x,t}^{y} + \mathbf{h}_{x,t+1}^{y}\mathbf{L}_{\mathbf{x}}^{\mathbf{x}}, \tag{D.30}$$

$$\mathbf{L}_{\mathbf{z}}^{\mathbf{y}} = \mathbf{h}_{z,t}^{y} + \mathbf{h}_{x,t+1}^{y}\ \mathbf{L}_{\mathbf{z}}^{\mathbf{x}}, \tag{D.31}$$

$$\mathbf{L}_{\mathbf{x}}^{\mathbf{c}} = \mathbf{h}_{x,t}^{c} + \mathbf{h}_{x,t+1}^{c}\mathbf{L}_{\mathbf{x}}^{\mathbf{x}}, \tag{D.32}$$

$$\mathbf{L}_{\mathbf{z}}^{\mathbf{c}} = \mathbf{h}_{z,t}^{c} + \mathbf{h}_{x,t+1}^{c}\ \mathbf{L}_{\mathbf{z}}^{\mathbf{x}}. \tag{D.33}$$

E Maximum-Likelihood Estimation

In this section of the appendix we examine the convergence behavior of the sequence $\{\mathbf{C}_t\}_{t=0}^N$ and provide a detailed the derivation of the conditional log-likelihood function presented in subsection 2.3.

E.1 Convergence behavior of the sequence $\{\mathbf{C}_t\}_{t=0}^N$

To elaborate the convergence behavior of the sequence $\{\mathbf{C}_t\}_{t=0}^N$, we first obtain two (well known) equivalent representations of the ricatti equation described by (9b).

Lemma E.1 *Suppose we define the matrices* $\bar{\mathbf{H}} \in \mathbb{R}^{n_y \times n_w}$, $\bar{\mathbf{G}} \in \mathbb{R}^{n_w \times n_y}$, $\bar{\mathbf{R}} \in \mathbb{R}^{n_y \times n_y}$, $\bar{\mathbf{F}} \in \mathbb{R}^{n_w \times n_w}$ *and* $\bar{\mathbf{Q}} \in \mathbb{R}^{n_w \times n_w}$ *as*

$$\bar{\mathbf{H}} := \mathbf{HF}, \qquad \bar{\mathbf{G}} := \mathbf{QH}^T, \qquad \bar{\mathbf{R}} := \mathbf{HQH}^T, \qquad \bar{\mathbf{F}} := \mathbf{F} - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}},$$

then we can rewrite (9b) *to*

$$\begin{aligned}
\mathbf{C}_t &= \mathbf{FC}_{t-1}\mathbf{F}^T + \mathbf{Q} - \left(\mathbf{HFC}_{t-1}\mathbf{F}^T + \mathbf{HQ}\right)^T \left[\mathbf{H}\left(\mathbf{FC}_{t-1}\mathbf{F}^T + \mathbf{Q}\right)\mathbf{H}^T\right]^{-1} \left(\mathbf{HFC}_{t-1}\mathbf{F}^T + \mathbf{HQ}\right) \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T - \bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{K}}_{t-1}^T\ \bar{\mathbf{F}}^T && \text{(LE.1a)} \\
&= \left[\bar{\mathbf{F}}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\ \bar{\mathbf{H}}\right)\right]\mathbf{C}_{t-1}\left[\bar{\mathbf{F}}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\ \bar{\mathbf{H}}\right)\right]^T + \left(\bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\right)\bar{\mathbf{R}}\left(\bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\right)^T && \text{(LE.1b)}
\end{aligned}$$

with $\bar{\mathbf{K}}_{t-1} := \mathbf{C}_{t-1}\bar{\mathbf{H}}^T\ \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1}$ *for all* $t = 1, 2, \ldots, N$.

Proof:
If we define

$$\bar{\mathbf{Q}} := \mathbf{Q} - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T,$$

we may state that

$$\begin{aligned}
\mathbf{C}_t &= \mathbf{FC}_{t-1}\mathbf{F}^T + \mathbf{Q} - \left(\mathbf{HFC}_{t-1}\mathbf{F}^T + \mathbf{HQ}\right)^T \left[\mathbf{H}\left(\mathbf{FC}_{t-1}\mathbf{F}^T + \mathbf{Q}\right)\mathbf{H}^T\right]^{-1} \left(\mathbf{HFC}_{t-1}\mathbf{F}^T + \mathbf{HQ}\right) \\
&= \mathbf{FC}_{t-1}\mathbf{F}^T + \mathbf{Q} - \left(\mathbf{FC}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{G}}\right)\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1}\left(\bar{\mathbf{H}}\mathbf{C}_{t-1}\mathbf{F}^T + \bar{\mathbf{G}}^T\right) \\
&= \left(\bar{\mathbf{F}} + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\right)\mathbf{C}_{t-1}\left(\bar{\mathbf{F}} + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\right)^T + \bar{\mathbf{Q}} + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&\qquad - \left[\left(\bar{\mathbf{F}} + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\right)\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{G}}\right]\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\left(\bar{\mathbf{F}} + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\right)^T + \bar{\mathbf{G}}^T\right] \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{Q}} + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&\qquad - \left[\bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{G}}\right]\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}^T\right]
\end{aligned}$$

$$
\begin{aligned}
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} - \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&\qquad + \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&\qquad - \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}^T\right] \\
&\qquad - \left[\bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{G}}\right] \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&\qquad - \left[\bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{G}}\right] \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}^T\right] \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} - \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&\qquad + \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&\qquad - \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&\qquad - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right] \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&\qquad - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right] \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} - \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&\qquad + \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&\qquad - \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} - \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T \left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} - \bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} - \bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1} \bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T \\
&= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T - \bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{K}}_{t-1}^T\,\bar{\mathbf{F}}^T + \bar{\mathbf{Q}}.
\end{aligned}
$$

Furthermore, since

$$
\begin{aligned}
&\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right)\mathbf{C}_{t-1}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right) + \bar{\mathbf{K}}_{t-1}\bar{\mathbf{R}}\bar{\mathbf{K}}_{t-1}^T \\
&= \left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right)\mathbf{C}_{t-1} - \left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right)\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\,\bar{\mathbf{K}}_{t-1}^T + \bar{\mathbf{K}}_{t-1}\bar{\mathbf{R}}\bar{\mathbf{K}}_{t-1}^T \\
&= \left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right)\mathbf{C}_{t-1} - \mathbf{C}_{t-1}\bar{\mathbf{H}}^T\,\bar{\mathbf{K}}_{t-1}^T + \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\,\bar{\mathbf{K}}_{t-1}^T + \bar{\mathbf{K}}_{t-1}\bar{\mathbf{R}}\bar{\mathbf{K}}_{t-1}^T \\
&= \left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right)\mathbf{C}_{t-1} - \mathbf{C}_{t-1}\bar{\mathbf{H}}^T\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\,\bar{\mathbf{K}}_{t-1}^T \\
&\qquad + \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T\,\bar{\mathbf{K}}_{t-1}^T + \bar{\mathbf{K}}_{t-1}\bar{\mathbf{R}}\bar{\mathbf{K}}_{t-1}^T \\
&= \left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right)\mathbf{C}_{t-1} - \bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\,\bar{\mathbf{K}}_{t-1}^T + \bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{K}}_{t-1}^T \\
&= \left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\,\bar{\mathbf{H}}\right)\mathbf{C}_{t-1}
\end{aligned}
$$

we can also state that

$$
\begin{aligned}
\mathbf{C}_t &= \bar{\mathbf{F}}\mathbf{C}_{t-1}\bar{\mathbf{F}}^T - \bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{K}}_{t-1}^T\ \bar{\mathbf{F}}^T + \bar{\mathbf{Q}} \\
&= \bar{\mathbf{F}}\left[\mathbf{C}_{t-1} - \bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\bar{\mathbf{K}}_{t-1}^T\ \right]\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} \\
&= \bar{\mathbf{F}}\left[\mathbf{C}_{t-1} - \bar{\mathbf{K}}_{t-1}\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]\left[\bar{\mathbf{H}}\mathbf{C}_{t-1}\bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right]^{-1}\bar{\mathbf{H}}\bar{\mathbf{C}}_{t-1}\ \right]\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} \\
&= \bar{\mathbf{F}}\left[\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\bar{\mathbf{H}}\right)\bar{\mathbf{C}}_{t-1}\right]\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} \\
&= \bar{\mathbf{F}}\left[\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\ \bar{\mathbf{H}}\right)\mathbf{C}_{t-1}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\ \bar{\mathbf{H}}\right) + \bar{\mathbf{K}}_{t-1}\bar{\mathbf{R}}\bar{\mathbf{K}}_{t-1}^T\right]\bar{\mathbf{F}}^T + \bar{\mathbf{Q}} \\
&= \left[\bar{\mathbf{F}}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\ \bar{\mathbf{H}}\right)\right]\mathbf{C}_{t-1}\left[\bar{\mathbf{F}}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\ \bar{\mathbf{H}}\right)\right]^T + \left(\bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\right)\bar{\mathbf{R}}\left(\bar{\mathbf{F}}\bar{\mathbf{K}}_{t-1}\right)^T + \bar{\mathbf{Q}}.
\end{aligned}
$$

Finally, (LE.1a) and (LE.1b) follow from the fact that

$$
\begin{aligned}
\bar{\mathbf{Q}} &= \mathbf{Q} - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{G}}^T \\
&= \mathbf{Q} - \mathbf{Q}\mathbf{H}^T\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\boldsymbol{\Sigma}\,\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right)^{-1}\mathbf{H}\mathbf{Q} \\
&= \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\\ \left(\mathbf{L}_{\mathbf{x}}^{\mathbf{y}}\right)^T\end{bmatrix}\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\boldsymbol{\Sigma}\,\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right)^{-1}\begin{bmatrix}\mathbf{L}_{\mathbf{z}}^{\mathbf{y}} & \mathbf{L}_{\mathbf{x}}^{\mathbf{y}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \\
&= \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{\Sigma}\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\\ \mathbf{0}\end{bmatrix}\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\boldsymbol{\Sigma}\,\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right)^{-1}\begin{bmatrix}\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\boldsymbol{\Sigma} & \mathbf{0}\end{bmatrix} \\
&= \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{\Sigma}\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\boldsymbol{\Sigma}\,\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right)^{-1}\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \\
&= \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{\Sigma}\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\left(\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^{-1}\right)^T\boldsymbol{\Sigma}^{-1}\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^{-1}\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \\
&= \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \\
&= \mathbf{0}.
\end{aligned}
$$

□

Using (LE.1b) it becomes obvious, that $\mathbf{C}^* = \mathbf{0}$ is a fixed point to the ricatti equation described by (9b), so that for $\mathbf{C}_0 = \mathbf{C}^*$ we may state that $\mathbf{C}_t = \mathbf{C}^*$ for all $t \in \mathbb{N}$. In the following proposition, we will further provide sufficient conditions under which the sequence $\{\mathbf{C}\}_{t=0}^N$ converges to this fixed point as t goes to infinity.

Proposition E.1 *Suppose the matrix* $\mathbf{C}_0 = \mathbf{C}_0^T \in \mathbb{R}^{n_w \times n_w}$ *with*

$$\mathbf{x}^T \mathbf{C}_0 \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^{n_w}, \mathbf{x} \neq \mathbf{0},$$

is an arbitrary initialization to the sequence $\{\mathbf{C}_t\}_{t=0}^N$ *determined by* (9b). *Further, suppose that all eigenvalues of the matrix* $\mathbf{Z}_2 = \mathbf{L}_\mathbf{x}^\mathbf{x} - \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1}\mathbf{L}_\mathbf{x}^\mathbf{y}$ *lie strictly within the unit circle. Then the sequence of matrices defined in eq. (9b) converges to* $\mathbf{C}^*$

$$\lim_{t \to \infty} \mathbf{C}_t = \mathbf{C}^*, \quad \textit{with } \mathbf{C}^* = \mathbf{0}.$$

Proof:

To proof Proposition E.1 we will first show that $\bar{\mathbf{F}}$ has the same set of non-zero eigenvalues as the matrix $\mathbf{L}_\mathbf{x}^\mathbf{x} - \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \mathbf{L}_\mathbf{x}^\mathbf{y}$. To see this, let us define the matrices $\mathbf{T} \in \mathbb{R}^{n_w \times n_w}$ and $\mathbf{Z} \in \mathbb{R}^{n_w \times n_w}$ as

$$\mathbf{T} := \begin{pmatrix} \mathbf{0} & -\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \mathbf{L}_\mathbf{x}^\mathbf{y} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \quad \text{and} \quad \mathbf{Z} := \begin{pmatrix} \mathbf{0} & \mathbf{Z}_1 \\ \mathbf{0} & \mathbf{Z}_2 \end{pmatrix},$$

with

$$\mathbf{Z}_1 := -\mathbf{\Pi}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \mathbf{L}_\mathbf{x}^\mathbf{y} \quad \text{and} \quad \mathbf{Z}_2 := \mathbf{L}_\mathbf{x}^\mathbf{x} - \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \mathbf{L}_\mathbf{x}^\mathbf{y}.$$

It follows from the definition of $\bar{\mathbf{F}}$ that

$$\begin{aligned}
\bar{\mathbf{F}} &= \mathbf{F} - \bar{\mathbf{G}}\bar{\mathbf{R}}^{-1}\bar{\mathbf{H}} \\
&= \mathbf{F} - \mathbf{Q}\mathbf{H}^T \left(\mathbf{L}_\mathbf{z}^\mathbf{y}\, \mathbf{\Sigma}\, \mathbf{L}_\mathbf{z}^{\mathbf{y}T}\right)^{-1} \mathbf{H}\mathbf{F} \\
&= \left(\mathbf{I} - \mathbf{Q}\mathbf{H}^T \left(\mathbf{\Sigma}\, \mathbf{L}_\mathbf{z}^{\mathbf{y}T}\right)^{-1} \left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \mathbf{H}\right)\mathbf{F} \\
&= \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{L}_\mathbf{z}^{\mathbf{y}T} \\ \mathbf{L}_\mathbf{x}^{\mathbf{y}T} \end{pmatrix} \left(\mathbf{\Sigma}\, \mathbf{L}_\mathbf{z}^{\mathbf{y}T}\right)^{-1} \left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \begin{pmatrix} \mathbf{L}_\mathbf{z}^\mathbf{y} & \mathbf{L}_\mathbf{x}^\mathbf{y} \end{pmatrix}\right)\mathbf{F} \\
&= \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{\Sigma}\mathbf{L}_\mathbf{z}^{\mathbf{y}T} \\ \mathbf{0} \end{pmatrix} \left(\mathbf{\Sigma}\, \mathbf{L}_\mathbf{z}^{\mathbf{y}T}\right)^{-1} \begin{pmatrix} \mathbf{I} & \left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \mathbf{L}_\mathbf{x}^\mathbf{y} \end{pmatrix}\right)\mathbf{F} \\
&= \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{I} & \left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1} \mathbf{L}_\mathbf{x}^\mathbf{y} \end{pmatrix}\right)\mathbf{F}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{I} & \left(\mathbf{L}_{\mathrm{z}}^{\mathrm{y}}\right)^{-1} \mathbf{L}_{\mathrm{x}}^{\mathrm{y}} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right) \mathbf{F} \\
&= \begin{pmatrix} \mathbf{0} & -\left(\mathbf{L}_{\mathrm{z}}^{\mathrm{y}}\right)^{-1} \mathbf{L}_{\mathrm{x}}^{\mathrm{y}} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \mathbf{F} \\
&= \mathbf{TF}.
\end{aligned}
$$

Furthermore we may write

$$
\begin{aligned}
\mathbf{FT} &= \begin{pmatrix} \mathbf{\Pi} & \mathbf{0} \\ \mathbf{L}_{\mathrm{z}}^{\mathrm{x}} & \mathbf{L}_{\mathrm{x}}^{\mathrm{x}} \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\left(\mathbf{L}_{\mathrm{z}}^{\mathrm{y}}\right)^{-1} \mathbf{L}_{\mathrm{x}}^{\mathrm{y}} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & -\mathbf{\Pi} \left(\mathbf{L}_{\mathrm{z}}^{\mathrm{y}}\right)^{-1} \mathbf{L}_{\mathrm{x}}^{\mathrm{y}} \\ \mathbf{0} & \mathbf{L}_{\mathrm{x}}^{\mathrm{x}} - \mathbf{L}_{\mathrm{z}}^{\mathrm{x}}\left(\mathbf{L}_{\mathrm{z}}^{\mathrm{y}}\right)^{-1} \mathbf{L}_{\mathrm{x}}^{\mathrm{y}} \end{pmatrix} \\
&= \mathbf{Z}.
\end{aligned}
$$

Since $\mathbf{T}$ and $\mathbf{F}$ are both square matrices, $\bar{\mathbf{F}} = \mathbf{TF}$ and $\mathbf{Z} = \mathbf{FT}$ share the same set of eigenvalues.[19] Further, since $\mathbf{Z}$ is an upper block-triangular matrix, where the diagonal-blocks are $\mathbf{0}$ and $\mathbf{Z}_2$, $\bar{\mathbf{F}}$ and $\mathbf{Z}$ share the same set of non-zeros eigenvalues as $\mathbf{Z}_2 = \mathbf{L}_{\mathrm{x}}^{\mathrm{x}} - \mathbf{L}_{\mathrm{z}}^{\mathrm{x}}\left(\mathbf{L}_{\mathrm{z}}^{\mathrm{y}}\right)^{-1} \mathbf{L}_{\mathrm{x}}^{\mathrm{y}}$.

In the next step we define a matrix sequence $\{\mathbf{C}_t^{(0)}\}_{t=1}^{N}$ with $\mathbf{C}_0^{(0)} = \mathbf{C}_0$ and

$$
\mathbf{C}_t^{(0)} = \bar{\mathbf{F}} \mathbf{C}_{t-1}^{(0)} \bar{\mathbf{F}}^T, \qquad \forall t = 1, 2 \dots, N.
$$

If we now can state for some $t = 1, 2, \dots, N$ that

$$
0 \leq \mathbf{x}^T \mathbf{C}_{t-1} \mathbf{x} \leq \mathbf{x}^T \mathbf{C}_{t-1}^{(0)} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^{n_w}, \mathbf{x} \neq \mathbf{0}, \tag{E.34}
$$

we may use Lemma E.1 to write

$$
\begin{aligned}
0 &\leq \mathbf{x}^T \left[\bar{\mathbf{F}}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\, \bar{\mathbf{H}}\right)\right] \mathbf{C}_{t-1} \left[\bar{\mathbf{F}}\left(\mathbf{I} - \bar{\mathbf{K}}_{t-1}\, \bar{\mathbf{H}}\right)\right]^T \mathbf{x} + \mathbf{x}^T \left(\bar{\mathbf{F}} \bar{\mathbf{K}}_{t-1}\right) \bar{\mathbf{R}} \left(\bar{\mathbf{F}} \bar{\mathbf{K}}_{t-1}\right)^T \mathbf{x} \\
&= \mathbf{x}^T \mathbf{C}_t \mathbf{x} \\
&= \mathbf{x}^T \bar{\mathbf{F}} \mathbf{C}_{t-1} \bar{\mathbf{F}}^T \mathbf{x} - \mathbf{x}^T \bar{\mathbf{F}} \bar{\mathbf{K}}_{t-1} \left[\bar{\mathbf{H}} \mathbf{C}_{t-1} \bar{\mathbf{H}}^T + \bar{\mathbf{R}}\right] \bar{\mathbf{K}}_{t-1}^T\, \bar{\mathbf{F}}^T \mathbf{x} \\
&\leq \mathbf{x}^T \bar{\mathbf{F}} \mathbf{C}_{t-1} \bar{\mathbf{F}}^T \mathbf{x} \leq \mathbf{x}^T \bar{\mathbf{F}} \mathbf{C}_{t-1}^{(0)} \bar{\mathbf{F}}^T \mathbf{x} = \mathbf{x}^T \mathbf{C}_t^{(0)} \mathbf{x}.
\end{aligned}
$$

[19] See Theorem 6.12 by Searle and Khuri (2017, pp. 140).

Thus, since (E.34) holds for $t = 1$ it follows inductively that

$$0 \leq \mathbf{x}^T \mathbf{C}_t \mathbf{x} \leq \mathbf{x}^T \mathbf{C}_t^{(0)} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^{n_w}, \mathbf{x} \neq \mathbf{0}, \forall t = 1, 2, \ldots, N.$$

Furthermore, we may state that

$$0 \leq \lim_{t\to\infty} \mathbf{x}^T \mathbf{C}_t \mathbf{x} \leq \lim_{t\to\infty} \mathbf{x}^T \mathbf{C}_t^{(0)} \mathbf{x} = \lim_{t\to\infty} \mathbf{x}^T \bar{\mathbf{F}}^t \ \mathbf{C}_0^{(0)} \left(\bar{\mathbf{F}}^t\right)^T \mathbf{x} = 0, \quad \mathbf{x} \in \mathbb{R}^{n_w}, \mathbf{x} \neq \mathbf{0},$$

since all eigenvalues of $\mathbf{Z}_2$ and therefore all eigenvalues of $\bar{\mathbf{F}}$ are strictly within the unit circle. This means that

$$\lim_{t\to\infty} \mathbf{x}^T \mathbf{C}_t \mathbf{x} = 0, \quad \forall \ \mathbf{x} \in \mathbb{R}^{n_w}, \mathbf{x} \neq \mathbf{0},$$

and thus that

$$\lim_{t\to\infty} \mathbf{C}_t = \mathbf{C}^* = \mathbf{0},$$

since $\mathbf{C}^* = \mathbf{0}$ is the only symmetric matrix satisfying

$$\mathbf{x}^T \mathbf{C}^* \mathbf{x} = 0, \quad \forall \ \mathbf{x} \in \mathbb{R}^{n_w}, \mathbf{x} \neq \mathbf{0}.$$

This completes the proof.[20]

□

Additionally we may show that under the preconditions of Propositions E.1 the solution $\mathbf{C}_* = \mathbf{0}$ is the unique stabilizing of the discrete ricatti equation (LE.1b). To see this, note that by definition any solution $\mathbf{C}_+$ to (LE.1b) is called the stabilizing solution to (LE.1b), if all eigenvalues of the matrix

$$\tilde{\mathbf{F}} = \bar{\mathbf{F}} \left(\mathbf{I} - \mathbf{C}_+ \bar{\mathbf{H}}^T \left[\bar{\mathbf{H}} \mathbf{C}_+ \bar{\mathbf{H}}^T + \bar{\mathbf{R}} \right]^{-1} \bar{\mathbf{H}} \right) \tag{E.35}$$

are strictly inside the unit circle. Thus, $\mathbf{C}_* = \mathbf{0}$ is a stabilizing solution to (LE.1b) if all eigenvalues of $\bar{\mathbf{F}}$ (or equivalently if all eigenvalues of $\mathbf{Z}_2$) are inside the unit circle. As shown by De Souza et al. (1986, Theorems 3.2 C), 4.1 A)) the existence of a stabilizing solution ensures an exponentially fast convergence to this solution. Further, as discussed

[20]Note that some arguments for this proof are borrowed from (Hamilton, 1994, Chapter 13).

by Harvey (1990, pp. 129) exponentially fast convergence is essential, since this ensures that the consistency of the ML estimator do not depend on the filter's initialization.

E.2 Derivation of the conditional log-likelihood L_C

To derive equations (13), (3a) and (3a) used for determining the conditional log-likelihood (i.e., $\mathbf{w}_0 = \mathbf{0}$ and $\mathbf{w}_t = \boldsymbol{\mu}_t,\ \forall t = 1, 2, \ldots, N$), recall that we assumed the matrix $\mathbf{L}_\mathbf{z}^\mathbf{y}$ to be non-singular. Thus (3a) and (3a) follow directly from (2a) and (2c) as

$$\begin{aligned}
\mathbf{z}_t &= \left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1}\left(\mathbf{y}_t - \mathbf{L}_\mathbf{x}^\mathbf{y}\,\mathbf{x}_t\right), \\
\mathbf{x}_t &= \mathbf{L}_\mathbf{z}^\mathbf{x}\mathbf{z}_{t-1} + \mathbf{L}_\mathbf{x}^\mathbf{x}\mathbf{x}_{t-1} \\
&= \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1}\left(\mathbf{y}_{t-1} - \mathbf{L}_\mathbf{x}^\mathbf{y}\mathbf{x}_{t-1}\right) + \mathbf{L}_\mathbf{x}^\mathbf{x}\mathbf{x}_{t-1} \\
&= \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1}\mathbf{y}_{t-1} + \left(\mathbf{L}_\mathbf{x}^\mathbf{x} - \mathbf{L}_\mathbf{z}^\mathbf{x}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^{-1}\mathbf{L}_\mathbf{x}^\mathbf{y}\right)\mathbf{x}_{t-1}, \quad \mathbf{x}_0 = \mathbf{0},\ \mathbf{y}_0 = \mathbf{H}\mathbf{w}_0 = \mathbf{0},\ \forall t = 1, 2, \ldots, N.
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
\mathbf{y}_t - \mathbf{H}\mathbf{F}\mathbf{w}_{t-1} = \mathbf{H}\mathbf{w}_t - \mathbf{H}\mathbf{F}\mathbf{w}_{t-1} = \mathbf{H}\left(\mathbf{w}_t - \mathbf{F}\mathbf{w}_{t-1}\right) = \mathbf{H}\mathbf{v}_t &= \begin{pmatrix}\mathbf{L}_\mathbf{z}^\mathbf{y} & \mathbf{L}_\mathbf{x}^\mathbf{y}\end{pmatrix}\begin{pmatrix}\boldsymbol{\epsilon}_t \\ \mathbf{0}\end{pmatrix} = \mathbf{L}_\mathbf{z}^\mathbf{y}\boldsymbol{\epsilon}_t \\
&= \mathbf{L}_\mathbf{z}^\mathbf{y}\left(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1}\right), \\
\mathbf{H}\mathbf{Q}\mathbf{H}^T &= \begin{pmatrix}\mathbf{L}_\mathbf{z}^\mathbf{y} & \mathbf{L}_\mathbf{x}^\mathbf{y}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^T \\ \left(\mathbf{L}_\mathbf{x}^\mathbf{y}\right)^T\end{pmatrix} = \begin{pmatrix}\mathbf{L}_\mathbf{z}^\mathbf{y}\boldsymbol{\Sigma} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^T \\ \left(\mathbf{L}_\mathbf{x}^\mathbf{y}\right)^T\end{pmatrix} \\
&= \mathbf{L}_\mathbf{z}^\mathbf{y}\boldsymbol{\Sigma}\left(\mathbf{L}_\mathbf{z}^\mathbf{y}\right)^T,
\end{aligned}$$

we receive the conditional log-likelihood

$$
\begin{aligned}
L_C &= \sum_{t=1}^{N} \ln \left|\mathbf{HQH}^T\right| + (\mathbf{y}_t - \mathbf{HFw}_{t-1})^T \left(\mathbf{HQH}^T\right)^{-1} (\mathbf{y}_t - \mathbf{HFw}_{t-1}) \\
&= \sum_{t=1}^{N} \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})\right)^T \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right)^{-1} \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})\right) \\
&= \sum_{t=1}^{N} \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right)^{-1}\ \mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1}) \\
&= \sum_{t=1}^{N} \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\ \boldsymbol{\Sigma}^{-1}\ (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1}) \\
&= N\ \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + \sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\ \boldsymbol{\Sigma}^{-1}\ (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1}) \\
&= N\ \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + \sum_{t=1}^{N} \operatorname{tr}\left((\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\ \ \boldsymbol{\Sigma}^{-1}\right) \\
&= N\ \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + \operatorname{tr}\left(\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\ \ \boldsymbol{\Sigma}^{-1}\right) \\
&= N\ \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + \operatorname{tr}\left(\left(\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\ \boldsymbol{\Sigma}^{-1}\right) \\
&= N\ \left[\ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + \operatorname{tr}\left(\left(\frac{1}{N}\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\boldsymbol{\Sigma}^{-1}\right)\right],
\end{aligned}
$$

as defined in (13). Finally, differentiation of (13) yields

$$
\begin{aligned}
\frac{\partial L_C}{\partial \boldsymbol{\Sigma}} &= N\ \frac{\partial \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + \operatorname{tr}\left(\left(\frac{1}{N}\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\boldsymbol{\Sigma}^{-1}\right)}{\partial \boldsymbol{\Sigma}} \\
&= N\ \frac{\partial \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\ \boldsymbol{\Sigma}\ \left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right|}{\partial \boldsymbol{\Sigma}} + N\ \frac{\partial \operatorname{tr}\left(\left(\frac{1}{N}\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\boldsymbol{\Sigma}^{-1}\right)}{\partial \boldsymbol{\Sigma}} \\
&= N\ \frac{\partial \ln \left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right|}{\partial \boldsymbol{\Sigma}} + N\ \frac{\partial \ln |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\Sigma}} + N\ \frac{\partial \ln \left|\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right|}{\partial \boldsymbol{\Sigma}} + N\ \frac{\partial \operatorname{tr}\left(\left(\frac{1}{N}\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\boldsymbol{\Sigma}^{-1}\right)}{\partial \boldsymbol{\Sigma}} \\
&= N\ \frac{\partial \ln |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\Sigma}} + N\ \frac{\partial \operatorname{tr}\left(\left(\frac{1}{N}\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\boldsymbol{\Sigma}^{-1}\right)}{\partial \boldsymbol{\Sigma}} \\
&= N\ \left(\boldsymbol{\Sigma}^T\right)^{-1} - N\ \left(\boldsymbol{\Sigma}\ \left(\frac{1}{N}\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\ \boldsymbol{\Sigma}\right)^T \\
&= N\ \boldsymbol{\Sigma}^{-1} - N\ \boldsymbol{\Sigma}\ \left(\frac{1}{N}\sum_{t=1}^{N} (\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T\right)\ \boldsymbol{\Sigma}\,. \qquad \text{(E.36)}
\end{aligned}
$$

From equating (E.36) to zero the conditional ML estimator for $\boldsymbol{\Sigma}$ follows as

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{t=1}^{N}(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})(\mathbf{z}_t - \boldsymbol{\Pi}\mathbf{z}_{t-1})^T .$$

Thus, L_C minimized with respect to $\boldsymbol{\Sigma}$ yields

$$L_{C,\hat{\boldsymbol{\Sigma}}} = N\left[\ln\left|\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\,\hat{\boldsymbol{\Sigma}}\,\left(\mathbf{L}_{\mathbf{z}}^{\mathbf{y}}\right)^T\right| + n_z\right].$$